In this paper, we introduce a new class of sets called $\tau^*$-generalized semiclosed sets and $\tau^*$-generalized semiopen sets in topological spaces and study some of their properties.

**KEYWORDS:** $\tau^*$ - gs closed set, $\tau^*$ - gs open set

**INTRODUCTION**


Throughout this paper X and Y are topological spaces on which no separation axioms are assumed unless otherwise explicitly stated. For a subset A of a topological space X, int(A), cl(A), cl$(\tau^*)$(A), scl(A), spcl(A), cl$_\alpha$(A), cl$_p$(A) and $A^c$ denote the interior, closure, $\tau^*$-closure, semi-closure, semi-preclosure, $\alpha$-closure, preclosure and complement of A respectively.

**PRELIMINARIES**

We recall the following definitions

**Definition: 2.1**

A subset A of a topological space $(X, \tau)$ is called

(i) Generalized closed (briefly g-closed)[6] if cl(A)$\subseteq G$ whenever $A \subseteq G$ and G is open in X.

(ii) Semi-generalized closed (briefly sg-closed)[3] if scl(A)$\subseteq G$ whenever $A \subseteq G$ and G is semi-open in X.

(iii) Generalized semi-closed (briefly gs-closed)[3] if scl(A)$\subseteq G$ whenever $A \subseteq G$ and G is semi-open in X.

(iv) $\alpha$-closed[8] if cl(int(cl(A))) $\subseteq A$

(v) $\alpha$-generalized closed (briefly ag-closed)[9] if cl$_\alpha$(A) $\subseteq G$ whenever $A \subseteq G$ and G is open in X.

(vi) Generalized $\alpha$-closed (briefly ga-closed)[10] if spcl(A) $\subseteq G$ whenever $A \subseteq G$ and G is open in X.
(vii) Generalized semi-preclosed (briefly gsp-closed)[2] if scl(A) ⊆ G whenever A ⊆ G and G is open in X.
(viii) Strongly generalized closed (briefly strongly g-closed)[12] if cl(A) ⊆ G whenever A ⊆ G and G is g-open in X.
(x) Semi-closed[7] if int(cl(A)) ⊆ A.
(xi) Semi-preclosed (briefly sp-closed)[1] if int(cl(int(A))) ⊆ A.
(xii) Generalized preclosed (briefly gp-closed)[13] if cl_p A ⊆ G whenever A ⊆ G and G is open.

The complements of the above mentioned sets are called their respective open sets.

Definition: 2.2
For the subset A of a topological X, the generalized closure operator cl' [5] is defined by the intersection of all g-closed sets containing A.

Definition: 2.3
For the subset A of a topological X, the topology τ* is defined by τ* = {G: cl'(G^c) = G^c}.

Definition: 2.4
For the subset A of a topological X,
(i) the semi-closure of \( A(briefly scl(A)) \)[7] is defined as the intersection of all semi-closed sets containing A.
(ii) the semi-Pre closure of \( A(briefly spcl(A)) \)[1] is defined as the intersection of all semi-preclosed sets containing A.
(iii) the \( \alpha - closure \) of \( A(briefly cl_\alpha(A)) \)[8] is defined as the intersection of all \( \alpha - closed \) sets containing A.
(iv) the preclosure of A, denoted by \( cl_p(A) \)[13], is the smallest preclosed set containing A.

Definition: 2.5
A subset A of a topological space X is called \( \tau^* \) generalized closed set (briefly \( \tau^* - gclosed \))[14] if cl'(A) ⊆ G whenever A ⊆ G and G is \( \tau^* - open \). The complement of \( \tau^* - generalized \) closed set is called the \( \tau^* - generalized \) open set (briefly \( \tau^* - gopen \)).

Definition: 2.6
A subset A of a topological space X is called \( \tau^* - generalized \) preclosed (briefly \( \tau^* - gp - closed \))[15] if cl'(cl_p(A)) ⊆ G whenever A ⊆ G and G is \( \tau^* - generalized \) open. The complement of \( \tau^* - generalized \) preclosed set is called the \( \tau^* - generalized \) preopen set (briefly \( \tau^* - gp - open \)).

\( \tau^* - generalized \) SEMICLOSED SETS IN TOPOLOGICAL SPACES
In this section, we introduce the concept of \( \tau^* - generalized \) semiclosed sets in topological spaces.

Definition: 3.1
A subset A of a topological space X is called \( \tau^* - generalized \) semiclosed (briefly \( \tau^* - gsclosed \)) if cl'(scl(A)) ⊆ G whenever A ⊆ G and G is \( \tau^* - open \). The complement of \( \tau^* - generalized \) semiclosed set is called the \( \tau^* - generalized \) semiopen set (briefly \( \tau^* - gopen \)).

Example: 3.2
Let \( X = \{a, b, c\} \) and let \( \tau = \{\phi, X, \{a\}, \{a, b\}, \{c\}, \{a, c\}\} \). Here \( (X, \tau^*) \) is \( \tau^* - generalized \) semiclosed

Theorem: 3.3
Every closed set in X is \( \tau^* - gclosed \).
Proof:
Let A be a closed set. Let A ⊆ G. Since A is closed, cl(A) = A ⊆ G. But cl'(scl(A)) ⊆ cl(A). Thus, we have cl'(scl(A)) ⊆ G whenever A ⊆ G and G is \( \tau^* - open \). Therefore A is \( \tau^* - gclosed \).

Theorem: 3.4
Every \( \tau^* - closed \) set in X is \( \tau^* - gclosed \).
Proof:
Let $A$ be a $\tau^* - closed$ set. Let $A \subseteq G$ where $G$ is $\tau^* - open$ since $A$ is $\tau^* - closed$, $cl'(scl(A)) = A \subseteq G$. Thus, we have $cl'(scl(A)) \subseteq G$ whenever $A \subseteq G$ and $G$ is $\tau^* - open$. Therefore $A$ is $\tau^* - gs$ closed.

**Theorem: 3.5**

Every $g$-closed set in $X$ is a $\tau^*$-gs-closed set but not conversely.

**Proof:**

Let $A$ be a $g$-closed set. Assume that $A \subseteq G$, $G$ is $\tau^* - open$ in $X$. Then $cl(A) \subseteq G$, Since $A$ is $g$-closed. But $cl'(scl(A)) \subseteq cl(A)$. Therefore $cl'(scl(A)) \subseteq G$. Hence $A$ is $\tau^* - gs$-closed.

The converse of the above theorem need not be true as seen from the following example.

**Example: 3.6**

Consider the topological space $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. Then, the set $\{a, c\}$ is $\tau^*-\text{gs} - closed$ but not $g$-closed.

**Remark: 3.7**

The following example shows that $\tau^* - gp - closed$ sets are independent from sp-closed, sg-closed set, pre-closed set, $gs$-closed set, $gsp$-closed set, $ag - closed$ set and $ga - closed$ set.

**Example: 3.8**

Let $X = \{a, b, c\}$ and $Y = \{a, b, c, d\}$ be the topological spaces.

(i) Consider the topology $\tau = \{X, \phi, \{a\}\}$. Then the sets $\{a\}, \{a, b\}$ and $\{a, c\}$ are $\tau^* - gs - closed$ but not sp-closed.

(ii) Consider the topology $\tau = \{X, \phi, \{a, b\}\}$. Then the sets $\{a\}$ and $\{b\}$ are sp-closed but not $\tau^* - gs - closed$.

(iii) Consider the topology $\tau = \{X, \phi\}$. Then the sets $\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}$ and $\{a, c\}$ are $\tau^* - gs - closed$ but not sg-closed.

(iv) Consider the topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the sets $\{a\}$ and $\{b\}$ are sg-closed but not $\tau^* - gs - closed$.

(v) Consider the topology $\tau = \{X, \phi, \{a\}\}$. Then the sets $\{a\}, \{b\}, \{c\}, \{a, b\}$ and $\{a, c\}$ are $\tau^* - gs - closed$ but not $\tau^* - closed$.

(vi) Consider the topology $\tau = \{X, \phi, \{a, b\}\}$. Then the set $\{b\}$ is $\alpha - closed$ but not $\tau^* - gs - closed$.

(vii) Consider the topology $\tau = \{X, \phi, \{a\}\}$. Then the sets $\{a\}, \{a, b\}$ and $\{a, c\}$ are $\tau^* - gs - closed$ but not pre-closed.

(viii) Consider the topology $\tau = \{X, \phi, \{a, b\}\}$. Then the set $\{a\}$ is pre-closed but not $\tau^* - gs - closed$.

(ix) Consider the topology $\tau = \{X, \phi\}$. Then the sets $\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}$ and $\{a, c\}$ are $\tau^* - gs - closed$ but not $gs$-closed.

(x) Consider the topology $\tau = \{Y, \phi, \{a, b, c\}, \{a, b, d\}\}$. Then the sets $\{b\}, \{c\}$ and $\{b, d\}$ are $gs - closed$ but not $\tau^* - gs - closed$.

(xi) Consider the topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the sets $\{b\}$ and $\{a, b\}$ are $gs - closed$ but not $\tau^* - gs - closed$.

(xii) Consider the topology $\tau = \{Y, \phi, \{a\}\}$. Then the set $\{a\}$ is $\tau^* - gs - closed$ but not $gs$-closed.

(xiii) Consider the topology $\tau = \{X, \phi, \{a\}\}$. Then the set $\{a\}$ is $\tau^* - gs - closed$ but not $ag$-closed.

(xiv) Consider the topology $\tau = \{Y, \phi, \{a, b, c\}, \{a, b, d\}\}$. Then the set $\{b\}, \{c\}, \{b, d\}$ are $ag - closed$ but not $\tau^* - gs - closed$.

(xv) Consider the topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the set $\{b\}$ is $\tau^* - gs - closed$ but not $ga - closed$.

(xvi) Consider the topology $\tau = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the set $\{b\}$, $\{b, c\}$ and $\{b, d\}$ are $ga - closed$ but not $\tau^* - gs - closed$.

**Theorem: 3.9**

For any two sets $A$ and $B$ $cl'(scl(A \cup B)) = cl'(scl(A)) \cup cl'(scl(B))$
Proof:
Since \( A \subseteq A \cup B \), we have \( cl'(scl(A)) \subseteq cl'(scl(A \cup B)) \) and Since \( B \subseteq A \cup B \), we have \( cl'(scl(B)) \subseteq cl'(scl(A \cup B)) \). Therefore \( cl'(scl(A)) \cup cl'(scl(B)) \subseteq cl'(scl(A \cup B)) \). Also, \( cl'(scl(A)) \) and \( cl'(scl(B)) \) are the closed sets. Therefore \( cl'(scl(A)) \cup cl'(scl(B)) \) is also a closed set. Again, \( A \subseteq cl'(scl(A)) \) and \( B \subseteq cl'(scl(B)) \), implies \( U \subseteq cl'(scl(A)) \cup cl'(scl(B)) \). Thus, \( cl'(scl(A)) \cup cl'(scl(B)) \) is a closed set containing \( A \cup B \). Since \( cl'(scl(A \cup B)) \) is the smallest closed set containing \( A \cup B \). We have \( cl'(scl(A \cup B)) \subseteq cl'(scl(A)) \cup cl'(scl(B)) \). Thus, \( cl'(scl(A \cup B)) = cl'(scl(A)) \cup cl'(scl(B)) \).

**Theorem 3.10**
Union of two \( \tau^* - gs\) closed sets in \( X \) is a \( \tau^* - gs\) closed set in \( X \).

**Proof:**
Let \( A \) and \( B \) be two \( \tau^* - gs\) closed sets. Let \( A \cup B \subseteq G \), where \( G \) is \( \tau^* - open \). Since \( A \) and \( B \) are \( \tau^* - gs \) closed sets, \( cl'(scl(A)) \cup cl'(scl(B)) \subseteq G \). But by theorem 3.9 \( cl'(scl(A)) \cup cl'(scl(B)) = cl'(scl(A \cup B)) \). Therefore \( cl'(scl(A \cup B)) \subseteq G \). Hence \( A \cup B \) is a \( \tau^* - gs\) closed set.

**Theorem 3.11**
A subset \( A \) of \( X \) is \( \tau^* - gs\) closed if and only if \( cl'(scl(A)) \) – \( A \) contains no non-empty \( \tau^* - closed \) set in \( X \).

**Proof:**
Let \( A \) be a \( \tau^* - gs\) closed set. Suppose that \( F \) is a non-empty \( \tau^* - closed \) subset of \( cl'(scl(A)) \) – \( A \). Now, \( F \subseteq cl'(scl(A)) \) – \( A \). Then \( F \subseteq cl'(scl(A)) \cap A^c \). Since \( cl'(scl(A)) \) – \( A = cl'(scl(A)) \cap A^c \). Therefore \( F \subseteq cl'(scl(A)) \) and \( F \subseteq A^c \). Since \( F^c \) is a \( \tau^*\)-open set and \( A \) is a \( \tau^* - gs\) closed, \( cl'(scl(A)) \subseteq F^c \). That is \( F \subseteq cl'(scl(A)) \cap [cl'(scl(A))]^c = \phi \). That is \( F = \phi \), a contradiction. Thus, \( cl'(scl(A)) \) – \( A \) contains no non-empty \( \tau^* - closed \) set in \( X \). Conversely, assume that \( cl'(scl(A)) \) – \( A \) contains no non-empty \( \tau^* - closed \) set. Let \( A \subseteq G \), \( G \) is \( \tau^* - open \). Suppose that \( cl'(scl(A)) \) is not contained in \( G \). then \( cl'(scl(A)) \cap G^c \) is a non-empty \( \tau^* - closed \) set of \( cl'(scl(A)) \) – \( A \) which is a contradiction. Therefore, \( cl'(scl(A)) \) – \( A \subseteq G \) and hence \( A \) is \( \tau^* - gs\) closed.

**Corollary 3.12**
A subset \( A \) of \( X \) is \( \tau^* - gs\) closed if and only if \( cl'(scl(A)) \) – \( A \) contains no non-empty \( \tau^* - closed \) set in \( X \).

**Proof:**
The proof follows from the theorem 3.11 and the fact that every closed set is \( \tau^* - closed \) set in \( X \).

**Corollary 3.13**
A subset \( A \) of \( X \) is \( \tau^* - gs \) closed if and only if \( cl'(scl(A)) \) – \( A \) contains no non-empty \( \tau^* - open \) set in \( X \).

**Proof:**
The proof follows from the theorem 3.11 and the fact that every open set is \( \tau^* - open \) set in \( X \).

**Theorem 3.14**
If a subset \( A \) of \( X \) is \( \tau^* - gs\) closed and \( A \subseteq B \subseteq cl'(scl(A)) \), then \( B \) is \( \tau^* - gs\) closed set in \( X \).

**Proof:**
Let \( A \) be a \( \tau^* - gs\) closed set such that \( A \subseteq B \subseteq cl'(scl(A)) \). Let \( U \) be a \( \tau^* - open \) set of \( X \) such that \( B \subseteq U \). Since \( A \) is \( \tau^* - gs\) closed, we have \( cl'(scl(A)) \subseteq U \).

Now, \( cl'(scl(A)) \subseteq cl'(scl(B)) \subseteq cl'(cl'(scl(A))) = cl'(scl(A)) \subseteq U \).

That is \( cl'(scl(B)) \subseteq U \). \( U \) is \( \tau^* - open \).

Therefore \( B \) is \( \tau^* - gs\) closed set in \( X \).

The converse of the above theorem need not be true as seen from the following example.

**Example 3.15**
Consider the topological space \( X = \{a, b, c\} \) with topology \( \tau = \{X, \phi, [a], [a, b]\} \). Then \( A \) and \( B \) are \( \tau^* - gs\) closed sets in \( (X, \tau) \). But \( A \subseteq B \) is not a subset of \( cl'(scl(A)) \).

**Theorem 3.16**
Let \( A \) be a \( \tau^* - gs\) closed in \( (X, \tau) \). Then \( A \) is \( g - closed \) if and only if \( cl'(scl(A)) \) – \( A \) is \( \tau^* - open \).

**Proof:**
Suppose $A$ is $g$-closed in $X$. Then, $\text{cl}'(\text{scl}(A)) = A$ and so $\text{cl}'(\text{scl}(A)) - A = \emptyset$ which is $\tau' - \text{open}$ in $X$. Conversely, suppose $\text{cl}'(\text{scl}(A)) - A$ is $\tau' - \text{open}$ in $X$. Since $A$ is $\tau' - \text{gs-closed}$, by the theorem 3.11, $\text{cl}'(\text{scl}(A)) - A$ contains no non-empty $\tau' - \text{closed}$ set in $X$. Then, $\text{cl}'(\text{scl}(A)) - A = \emptyset$. Hence, $A$ is $g$-closed.

**Remark 3.17**

From the above discussion, we obtain the following implications.

**REFERENCES**


