In this paper a comparison of some biased reduced order models is carried out. These methods use the advantages of Padé approximation, stability equation method, continued fraction expansion method based on first and second Cauer forms, factor division algorithm, and Routh approximation. One numerical example illustrates the method.

KEYWORDS: Stability, Biased Models, Model Reduction.

INTRODUCTION
Model order reduction means to reduce a high order system into a low order model which retains the desired characteristics of the system. A large number of research papers on model order reduction have been published [1-4]. Some parameters are changed to get various reduced order models. In this paper four methods are discussed to produce reduced order models. The first method is based on stability equation method and a combination of time moments and Markov parameters [5-6]. The second method utilises the advantages of continued fraction expansion and time moments and Markov parameters [7]. In the third method factor division is coupled with time moments and Markov parameters [8-9]. Fourth Simplified Routh Approximation Method is coupled with the combination of time moments and Markov parameters [10]. Some biased methods are extended for order reduction of multivariable system as in Improved Padé Approximants Using Stability Equation is extended for multivariable system. In this paper comparison of biased reduced models is carried out.

STATEMENT OF PROBLEM
Let higher order system may be described by the transfer function

\[ G(s) = \frac{A_{21} + A_{22} s + A_{23} s^2 + \cdots + A_{2n} s^{n-1}}{A_{11} + A_{12} s + A_{13} s^2 + \cdots + A_{1n} + s^n} \]  

(1)

Objective is to reduce the above \( n \)th order system to \( r \)th order reduced model which is defined by the transfer function

\[ R(s) = \frac{a_{21} + a_{22} s + a_{23} s^2 + \cdots + a_{2r} s^{r-1}}{a_{11} + a_{12} s + a_{13} s^2 + \cdots + a_{1r} + s^r} \]  

(2)

Where \( r \leq n \) and \( a_{i,j} \) are scalar constants. Such that \( R(s) \) retains the important properties of \( G(s) \).

REDUCTION METHOD
The \( r \)th-order reduced approximant \( R(s) \) for \( G(s) \) is obtained by different methods as
The reciprocal polynomial of $D(s)$ be defined by

$$\tilde{D}(s) = s^n D(\frac{1}{s})$$

$$= A_{11}s^n + A_{12}s^{n-1} + \ldots \ldots + A_{1,n}s + A_{1,n+1} \ldots (3)$$

The reciprocal polynomial in eqn.3 has the property that it inverts the roots of the original polynomial and thus the small magnitude roots of $D(s)$ will become the large magnitude roots of $D(s)$ and vice versa. For stable $G(s)$, the even and odd parts of $D(s)$ may be factored as the following stability equations: [5]

$$D_e(s) = A_{11} \prod_{i=1}^{k_1} (1 + \frac{s^2}{z_i^2}) \ldots \ldots \ldots (4(a))$$

$$D_o(s) = A_{12}s \prod_{i=1}^{k_2} (1 + \frac{s^2}{p_i^2}) \ldots \ldots \ldots (4(b))$$

Where $k_1$ and $k_2$ are the integer part of $n/2$ and $(n - 1)/2$ respectively and $z_1 < p_1 < z_2 < p_2 \ldots$. By discarding the factors with larger magnitudes of $z_i$ and $p_i$ we have a reduced stability equation of order $r_1$ as

$$D_{r_1}(s) = A_{11} \prod_{i=1}^{r_1} (1 + \frac{s^2}{z_i^2}) + A_{12}s \prod_{i=1}^{r_1-1} (1 + \frac{s^2}{p_i^2}) \ldots \ldots (5)$$

As discussed above, only poles nearest to the origin are retained in $D_{r_1}(s)$ and no consideration is given to the poles which have large negative real parts. To ensure that $D_{r_1}(s)$ also approximates some large magnitude poles of $G(s)$ stability equations similar to eqn. 4 are constructed for the reciprocal polynomial of eqn. 3 and a reduced polynomial $\tilde{D}_{r_2}(s)$ of order $r_2$ is formed. $D_r(s)$ is then found as

$$D_r(s) = D_{r_1}(s).D_{r_2}(s) \ldots \ldots (6)$$

Where $D_{r_2}(s)$ is the reciprocal polynomial of $\tilde{D}_{r_2}(s)$ and $r = r_1 + r_2$.

Assuming that $R(s)$ and $G(s)$ have identical first $a$ time moments and first $\beta$ Markov parameters, the coefficients of numerator of reduced order model are then determined from [6]

$$a_{2,1} = a_{11}T_0$$

$$a_{2,2} = a_{11}T_1 + a_{12}T_0$$

$$\ldots$$

$$a_{2,\alpha} = a_{11}T_{\alpha-1} + a_{12}T_{\alpha-2} + \ldots \ldots + a_{1,\alpha-1}T_1 + a_{1,\alpha}T_0$$

$$a_{2,(r-\beta+1)} = a_{1,r+1}M_{\beta-1} + a_{1,r}M_{\beta-2} + \ldots \ldots + a_{1,(r-\beta+3)}M_1 + a_{1,(r-\beta+2)}M_0$$

$$\ldots$$
\[ a_{2r-1} = a_{1r+1}M_1 + a_{1r}M_0 \]
\[ a_{2r} = a_{1r+1}M_0 \]

Where \( \alpha + \beta = r \).

**CONTINUED FRACTION ALGORITHM**

It is well known that the first \( t \) time moments and \((2r - t)\) Markov parameters of \( G(s) \) are retained in the reduced model formed by truncating the continued-fraction expansion after \( 2r \) quotients, the first \( t \) quotients being formed by division from the constant terms and the next \((2r - t)\) quotients by division from the highest powers of \( s \). This gives the continued-fraction in the form

\[
G(s) = \frac{1}{h_1 + \frac{s}{h_2 + \frac{s}{h_3 + \frac{s}{h_4 + \cdots}}}} \quad \cdots (7)
\]

The \( h_i (i = 1, 2 \ldots 2r) \) are readily obtained from the following **Routh-type array:**

| \( h_1 \) | \( a_1 \) | \( A_{11} \) | \( A_{12} \) | \( A_{13} \) | \( \ldots \) | \( A_{1, n+1} \) |
| \( h_2 \) | \( a_2 \) | \( A_{21} \) | \( A_{22} \) | \( A_{23} \) | \( \ldots \) | \( A_{2, n} \) |
| \( \vdots \) | \( a_n \) | \( A_{n1} \) | \( A_{n2} \) | \( A_{n3} \) | \( \ldots \) | \( A_{n, n} \) |

\[
h_{t+1} = \frac{Q_{t+1,1} + A_{t+1, 1} A_{t+2, 2} \ldots A_{t, n+1-(t+2)/2}}{B_{t+1, 1} B_{t+2, 2} \ldots A_{t+1, n+1-(t+2)/2}}
\]

\[
h_{t+2} = \frac{Q_{t+2, 2} + A_{t+2, 2} A_{t+3, 3} \ldots A_{t+1, n+1-(t+3)/2}}{B_{t+2, 2} B_{t+3, 3} \ldots A_{t+2, n+1-(t+3)/2}}
\]

\[
h_{2r-1} = \frac{Q_{2r-1, 1} + A_{2r-1, 1} A_{2r-2, 2} \ldots}{B_{2r-1, 1} B_{2r-2, 2} \ldots}
\]

\[
h_{2r} = \frac{Q_{2r, 2} + A_{2r, 2} A_{2r+1, 3} \ldots}{B_{2r, 2} B_{2r+1, 3} \ldots}
\]

Where

\[
\{ h_i = A_{i, i}/A_{i+1, 1} \quad i = 1, 2 \ldots t \}
\]
\[B_{i,j} = B_{i-2,j+1} - h_{i-2}B_{i-1,j+1}
\]
\[i = t + 1, t + 2, \quad j = 1, 2, \ldots, n + 1 - \left\lfloor \frac{i}{2} \right\rfloor\]

And the last two rows of the \(A_{i,j}\) form the first two rows of the \(B_{i,j}\) by reversing the order of the elements, i.e.

\[B_{i,j} = A_{i,n+2-\left\lfloor \frac{i}{2} \right\rfloor-1}
\]
\[i = t + 1, t + 2, \quad j = 1, 2, \ldots, n + 1 - \left\lfloor \frac{i}{2} \right\rfloor\]

To invert the continued fraction (eqn. 7) an inverse array is formed with the same \(h_{i}(i = 1, 2 \ldots 2r)\) starting from the bottom element \(Q_{2r+1,1} = 1\) (as this is arbitrary) which has the structure:

With

\[Q_{2r+1,1} = 1, Q_{1,1} = h_{1}Q_{1+1,1}
\]
\[i = 2r, 2r - 1, \ldots, t + 1\]

\[a_{i,1} = h_{i}a_{i+1,1} \quad i = t, t - 1, \ldots, 1\]

\[Q_{i-2,j+1} = Q_{i,j} + h_{i-2}Q_{i-1,j+1}
\]
\[i = 2r + 1, \ldots, t + 3, \quad j = 1, 2, \ldots, r + 1 - \left\lfloor \frac{1}{2} \right\rfloor\]

\[a_{i-2,j+1} = a_{i,j} + h_{i-2}a_{i-1,j+1}
\]
\[i = t + 2, \ldots, 3 \quad j = 1, 2, \ldots, r + 1 - \left\lfloor \frac{i}{2} \right\rfloor\]

And

\[a_{i,j} = Q_{(r+2-\left\lfloor \frac{i}{2} \right\rfloor)j}
\]
\[i = t + 1, t + 2, \quad j = 1, 2, \ldots, r + 1 - \left\lfloor \frac{i}{2} \right\rfloor\]

The reduced transfer function is then given by the first two rows of the array (eqn.8).
BIASEDMODEL REDUCTION BY FACTOR DIVISION

A reduced $r^{th}$-order model is to be found such that

$$ R(s) = \frac{N_r(s)}{D_r(s)} \quad \ldots \ldots \quad (9) $$

where $D_r(s)$ is the reduced stable denominator which may be found by one of the many techniques available and $N_r(s)$, the reduced numerator, is formed so that $R(s)$ retains the first $t$ time moments and $m$ Markov parameters of $G(s)$, where $t + m = r$.

Consider the polynomials defined by

$$ D(s) = A_{11} + A_{12}s + \ldots + A_{1,n+1}s^n $$

$$ N_t(s) = A_{21} + A_{22}s + \ldots + A_{2,t-1}s^{t-1} $$

$$ N_m(s) = A_{2,n-1}s^{n-1} + A_{2,n-2}s^{n-2} + \ldots + A_{2,n-m}s^{n-m} $$

$$ N_n(s) = A_{1,t}s^t + A_{1,t+1}s^{t+1} + \ldots + A_{1,n-m-1}s^{n-m-1} $$

Then from eqn.1

$$ G(s) = \frac{N_t(s)}{D(s)} + \frac{N_m(s)}{D(s)} + \frac{n(s)}{D(s)} \quad \ldots \ldots \quad (10) $$

It is clear that only the expressions $N_t(s)/D(s)$ and $N_m(s)/D(s)$ contribute to the first $t$ time moments and $m$ Markov parameters, respectively, to be retained in the reduced model. Thus $n(s)/D(s)$ may effectively be ignored for reduction purposes and eqn.10 is written as

$$ G(s) = \frac{N_t(s)D_r(s)}{D_r(s)} + \frac{N_m(s)D_r(s)}{D_r(s)} $$

The reduced model is then given by

$$ R(s) = \frac{\alpha_0 + \alpha_1s + \ldots + \alpha_{t-1}s^{t-1}}{D_r(s)} + \frac{\beta_1s^t + \ldots + \beta_1s^t}{D_r(s)} $$

$$ = \frac{\alpha_0 + \alpha_1s + \ldots + \alpha_{t-1}s^{t-1} + \beta_1s^t + \ldots + \beta_1s^t}{D_r(s)} \quad \ldots \ldots \quad (11) $$

Where $\frac{N_t(s)D_r(s)}{D(s)} = \alpha_0 + \alpha_1s + \ldots + \alpha_{t-1}s^{t-1} \ldots \ldots$ is calculated by the factor division algorithm [8] and
\[
\frac{N_m(s)D_r(s)}{D(s)} = \beta_m s^{r-1} + \beta_{m-1} s^{r-2} + \ldots + \beta_1 s^I + \ldots
\]

Is calculated using division from the highest powers of \(s\) by a similar procedure, description is given in [9].

**SIMPLIFIED ROUITH APPROXIMATION METHOD**

Let \(A_{21}=B_0, A_{22}=B_1, \ldots, A_{2n}=B_{n-1}\)

and \(A_{11}=A_0, A_{12}=A_1, \ldots, A_{1, n+1}=A_n\)

By using SRAM, the reduced denominator of \(r^{th}\) order model is defined as

\[
D_r(s) = s^r + \frac{\alpha_1}{A_0} \sum_{j=0}^{r-1} A_j s^j \ldots (12)
\]

The \(\alpha\)-parameters (\(\alpha_1, \alpha_2, \alpha_r\)) are obtained from the following simple Routh-type array:

**Alpha (a) table**

\[
\begin{align*}
\alpha_1 &= \frac{A_0}{A_1} < \begin{array}{cccccc}
A_0 & A_1 & A_2 & A_3 & A_4 & \ldots & A_n \\
A_1 & A_2 & A_3 & A_4 & \ldots & A_n
\end{array} \\
\alpha_2 &= \frac{C_1}{C_2} < \begin{array}{cccc}
C_1 & C_2 & C_3 & C_4 & \ldots & C_n \\
C_2 & C_3 & C_4 & C_5 & \ldots & C_n
\end{array} \\
\alpha_3 &= \frac{D_1}{D_2} < \begin{array}{cccc}
D_1 & D_2 & D_3 & D_4 & \ldots & D_{n-1} \\
D_2 & D_3 & D_4 & D_5 & \ldots & D_{n-1}
\end{array} \\
\alpha_4 &= \frac{E_1}{E_2} < \begin{array}{cccc}
E_1 & E_2 & E_3 & E_4 & \ldots & E_{n-2} \\
E_2 & E_3 & E_4 & E_5 & \ldots & E_{n-2}
\end{array}
\end{align*}
\]

For \(i = \) odd

\[
\begin{align*}
C_i &= A_i & i &= 1, 3, \ldots \ldots \\
D_1 &= C_2 \\
D_i &= A_{i+1} & i &= 3, 5, \ldots \ldots \\
E_1 &= D_2 \\
E_i &= A_{i+2} & i &= 3, 5, \ldots \ldots
\end{align*}
\]
For i = even
\[ C_i = A_i - \frac{A_0 A_{i+1}}{A_1} \]
\[ D_i = C_{i+1} - \frac{C_i C_{i+2}}{C_2} \]
\[ E_i = D_{i+1} - \frac{D_i D_{i+2}}{D_2} \]

Then for any given \( D_r(s) \), the numerator \( N_r(s) \) of the biased model, which will retain the first t time moments and m Markov parameters of \( G(s) \), is defined as

\[ N_r(s) = N_{rt}(s) + N_{rm}(s) \quad \text{with } r = t + m \]

\[ = T_1 + T_2 s + \cdots + T_4 s^{r-m+1} + M_m s^{r-m} + \cdots + M_2 s^{r-2} + M_1 s^{r-1} \]

In general \( T_t = \frac{a_{t+1}}{A_0} B_{t-1} \)

And

\[ M_m = \frac{1}{A_1} \left( \sum_{i=1}^{m} B_{n-m} a_{1,r+1-(m-j)} - \sum_{j=0}^{m-1} M_j A_{n-(m-j)} \right) \]

Where \( M_0 = 0 \)

EXAMPLE

Consider a 3rd order system also used in literature [6-7] which is reduced to the second order model by the above four methods

\[ G(s) = \frac{8s^2 + 6s + 2}{s^2 + 4s^2 + 5s + 2} \]

By Applying Improved PadeApproximation Using Stability equation various second order reduced models are obtained as:

\[ \alpha = \beta = 1, r_2 = 2: R_1(s) = \frac{8s + 5}{s^2 + 4s + 5} \]
\[ \alpha = 2, r_2 = 2: R_2(s) = \frac{6.5s + 5}{s^2 + 4s + 5} \]
\[ \alpha = 2, r_1 = r_2 = 1: R_3(s) = \frac{5.2s + 1.6}{s^2 + 4s + 5} \]
\[ \alpha_1 = 2, r_1 = 2: \quad R_4(s) = \frac{1.5s + 0.5}{s^2 + 1.25s + 0.5} \]

Comparison of step responses of full system and reduced models is shown in figure 1.

Applying continued fraction algorithm for biased model reduction we get two second order reduced model as

\[ m = 1, t = 3: \quad R_5(s) = \frac{8s + 3.8976}{s^2 + 3.7338s + 3.8976} \]

\[ m = 2, t = 2: \quad R_6(s) = \frac{8s + 7.6}{s^2 + 4.2s + 7.6} \]

Comparison of step responses of full system and reduced models is shown in figure 2.

Biased model reduction by factor division: Here denominator is obtained by Improved Pade Approximation Using Stability Equation as

\[ D(s) = 4s^2 + 5s + 2 \]

And \( \alpha, \beta \) parameters are given as

\[ \alpha_1 = 2, \alpha_2 = 6, \beta_1 = 32 \]

Now two second order models are obtained as

\[ m = 0, t = 2: \quad R_7(s) = \frac{6s + 2}{4s^2 + 5s + 2} \]

\[ m = 1, t = 1: \quad R_8(s) = \frac{32s + 2}{4s^2 + 5s + 2} \]

Comparison of step responses of full system and reduced models is shown in figure 3.

By SRAM: From \( \alpha \)-table \( \alpha_1 = 0.4 \) and \( \alpha_2 = 1.388889 \) and second order reduced denominator applying SRAM is obtained as

\[ D(s) = s^2 + \frac{\alpha_1 \alpha_2}{A_0}(A_0 + A_1s) \]

\[ = 0.5555556 + 1.3888889s + s^2 \]

And \( T_1 = 0.5555556, T_2 = 1.6666668 \) and \( M_1 = 8 \)

Two stable reduced models which retains \( t \) time moments and \( m \) Markov parameters are obtained from \( G(s) \), where \( t+m = 2 \)

\[ m = 0, t = 2: \quad R_9(s) = \frac{1.6666668s + 0.5555556}{s^2 + 1.3888889s + 0.5555556} \]

\[ m = 1, t = 1: \quad R_{10}(s) = \frac{8s + 0.5555556}{s^2 + 1.3888889s + 0.5555556} \]
Comparison of step responses of full system and reduced models is shown in figure 4.
CONCLUSIONS

It is shown that in model reduction by Pade Approximation, \( R_1(s) \) shows the best overall time response approximation to \( G(s) \). In model reduction by Continued Fraction, \( R_5(s) \) shows the best overall time response approximation to \( G(s) \). In model reduction by Factor Division, \( R_7(s) \) shows the best overall time response approximation to \( G(s) \). In model reduction by SRAM, \( R_8(s) \) shows the best overall time response approximation to \( G(s) \). The comparison of best responses of all the four methods and original system is shown in Fig. 5. And it is shown that model reduction by continued fraction \( R_5(s) \) gives the best response. SRAM is the simplest method of model reduction because only one simple Routh-type array can be used to generate denominator as well as numerator of the reduced-order stable biased model. Table-1 gives the comparison on the basis of various parameters.
TABLE-1

<table>
<thead>
<tr>
<th>System Characteristics</th>
<th>G(s)</th>
<th>R1(s)</th>
<th>R2(s)</th>
<th>R3(s)</th>
<th>R4(s)</th>
<th>R5(s)</th>
<th>R6(s)</th>
<th>R7(s)</th>
<th>R8(s)</th>
<th>R9(s)</th>
<th>R10(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rise time (sec)</td>
<td>0.129</td>
<td>0.131</td>
<td>0.171</td>
<td>0.284</td>
<td>0.856</td>
<td>0.128</td>
<td>0.13</td>
<td>0.836</td>
<td>0.109</td>
<td>0.765</td>
<td>0.11</td>
</tr>
<tr>
<td>Settling time (sec)</td>
<td>6.74</td>
<td>2.63</td>
<td>2.56</td>
<td>7.03</td>
<td>7.63</td>
<td>3.26</td>
<td>1.78</td>
<td>7.63</td>
<td>8.01</td>
<td>7.5</td>
<td>8.29</td>
</tr>
<tr>
<td>Peak Response</td>
<td>1.87</td>
<td>1.73</td>
<td>1.53</td>
<td>1.2</td>
<td>1.27</td>
<td>1.88</td>
<td>1.69</td>
<td>1.27</td>
<td>4.82</td>
<td>1.26</td>
<td>4.44</td>
</tr>
<tr>
<td>Amplitude Overshoot (%)</td>
<td>86.5</td>
<td>77.3</td>
<td>52.7</td>
<td>19.9</td>
<td>26.6</td>
<td>88</td>
<td>69.1</td>
<td>26.6</td>
<td>382</td>
<td>26</td>
<td>344</td>
</tr>
<tr>
<td>At time (sec)</td>
<td>0.656</td>
<td>0.63</td>
<td>0.682</td>
<td>1.02</td>
<td>2.6</td>
<td>0.674</td>
<td>0.55</td>
<td>2.6</td>
<td>1.59</td>
<td>2.34</td>
<td>1.51</td>
</tr>
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REFERENCES