The purpose of this paper is to study Lorentzian special Sasakian manifolds and generalized Lorentzian Co-symplectic manifolds [1] with semi-symmetric metric connection [3].

KEYWORDS: Nearly and almost LS-Sasakian manifolds, generalized L-Co-symplectic manifolds, semi-symmetric metric connection, Nijenhuis tensor [2].

INTRODUCTION

An n-dimensional differentiable manifold $M_n$, on which there are defined a tensor field $F$ of type $(1, 1)$, a vector field $T$, a 1-form $A$ and a Lorentzian metric $g$, satisfying for arbitrary vector fields $X, Y, Z, ...$

(1.1) $\bar{X} = -X - A(X)T, \quad \bar{T} = 0, \quad A(T) = -1, \quad \bar{X} \equiv FX, \quad A(\bar{X}) = 0, \quad \text{rank } F = n - 1.$

(1.2) $g(\bar{X}, \bar{Y}) = g(X, Y) + A(X)A(Y), \quad \text{where } A(X) = g(X, T),$ $\quad \bar{F}(X, Y) \equiv g(\bar{X}, Y) = -\bar{F}(Y, X).$

Then $M_n$ is called a Lorentzian contact manifold (an L-Contact manifold).

Let $D$ be a Riemannian connection on $M_n$, then An L-Contact manifold is called a Lorentzian special Sasakian manifold (an LS-Sasakian manifold), if

(1.3) (a) $(D_XF)(Y) + A(Y)\bar{X} - \bar{F}(X, Y)T = 0 \Leftrightarrow (D_X^\gamma F)(Y, Z) - A(Y)F(Z, X) - A(Z)F(X, Y) = 0$

(b) $D_XT = \bar{X}$

An L-Contact manifold is called a nearly Lorentzian special Sasakian manifold (a nearly LS-Sasakian manifold), if

(1.4) $(D_X^\gamma F)(Y, Z) - A(Y)F(Z, X) - A(Z)\bar{F}(X, Y)$

$= (D_Y^\gamma F)(Z, X) - A(Z)F(Y, X) - A(X)\bar{F}(Y, Z)$

$= (D_Y^\gamma F)(X, Y) - A(X)\bar{F}(Y, Z) - A(Y)\bar{F}(X, Z)$

An L-Contact manifold is called an almost Lorentzian special Sasakian manifold (an almost LS-Sasakian manifold), if

(1.5) $(D_X^\gamma F)(Y, Z) + (D_Y^\gamma F)(Z, X) + (D_Z^\gamma F)(X, Y) - 2(A(X)\bar{F}(Y, Z) + A(Y)\bar{F}(Z, X) + A(Z)\bar{F}(X, Y)) = 0$

An L-Contact manifold is called a generalized Lorentzian Co-symplectic manifold (a generalized L-Co-symplectic manifold), if

(1.6) (a) $(D_X^\gamma F)(Y) - A(Y)D_XF - (D_XA)(\bar{Y})T = 0 \Leftrightarrow$

(b) $(D_X^\gamma F)(Y, Z) + A(Y)(D_XA)(\bar{Z}) - A(Z)(D_XA)(\bar{Y}) = 0$

An L-Contact manifold is called a generalized nearly Lorentzian Co-symplectic manifold (a generalized nearly L-Co-symplectic manifold), if

(1.7) $(D_X^\gamma F)(Y, Z) + A(Y)(D_XA)(\bar{Z}) - A(Z)(D_XA)(\bar{Y})$

$= (D_Y^\gamma F)(Z, X) + A(Z)(D_YA)(\bar{X}) - A(X)(D_YA)(\bar{Z})$

$= (D_Z^\gamma F)(X, Y) + A(X)(D_ZA)(\bar{Y}) - A(Y)(D_ZA)(\bar{X})$

An L-Contact manifold is called a generalized almost L-Co-symplectic manifold, if

(1.8) $(D_X^\gamma F)(Y, Z) + (D_Y^\gamma F)(Z, X) + (D_Z^\gamma F)(X, Y) - A(X)\{(D_XA)(\bar{Z}) - (D_ZA)(\bar{Y})\}$

$- A(Y)\{(D_ZA)(\bar{X}) - (D_YA)(\bar{Z})\} - A(Z)\{(D_XA)(\bar{Y}) - (D_YA)(\bar{X})\} = 0$
Semi-Symmetric Metric Connection

Let us consider a connection $B$ on $M_{n}$, defined by

(2.1) $B_{X}Y \equiv D_{X}Y + A(Y)X - g(X,Y)T$

The torsion tensor $S$ of $B$ is given by

(2.2) $S(X,Y) = A(Y)X - A(X)Y$

Further, if $(B_{X}g) = 0$, then $B$ is called a semi-symmetric metric connection.

Put

(2.3) (a) $B_{X}Y = D_{X}Y + H(X,Y)$

Where $H$ is a tensor field of type $(1, 2)$, then

(b) $H(X,Y) = A(Y)X - g(X,Y)T$

(c) $H(X,Y,Z) = A(Y)g(Z,X) - A(Z)g(X,Y)$

(d) $S(X,Y,Z) = H(X,Y,Z) - H(Y,X,Z)$

Where

$H(X,Y,Z) \equiv g(H(X,Y),Z)$ and $S(X,Y,Z) \equiv g(S(X,Y),Z)$

In an L-Contact manifold, we have

(2.4) $(B_{X}F)(Y,Z) + (B_{X}F)(Z,Y) + A(Y)(B_{X}A)(Z) - A(Z)(B_{X}A)(Y) = 0$

Therefore,

An L-Contact manifold is called an LS-Sasakian manifold, if

(2.5) (a) $(B_{X}F)(Y,Z) - 2A(Y)F(X,Z) - 2A(Z)F(X,Y) = 0$

(b) $B_{X}T = 2 \overline{Y}$

On this manifold, we have

(2.6) (a) $(B_{X}A)(Y) = 2F(X,Y) \iff (b) (B_{X}A)(Y) = -2g(X,Y)$

An L-contact manifold is called a nearly LS-Sasakian manifold, if

(2.7) $(B_{X}F)(Y,Z) - 2A(Y)F(X,Z) - 2A(Z)F(X,Y) = 0$

These equations can be modified as

(2.8) (a) $(B_{X}F)Y + (B_{X}F)X + 2A(X)Y + 2A(X)Y = 0 \iff (b) (B_{X}F)(Y,Z) - 2A(Y)F(X,Z) + 2A(X)F(Y,Z) = 0$

(2.9) (a) $(B_{X}F)\overline{Y} + (B_{X}F)\overline{X} + 2A(X)\overline{Y} = 0 \iff (b) (B_{X}F)(\overline{Y},Z) - (B_{X}F)(Z,X) - 2A(X)F(Y,Z) = 0$

(2.10) (a) $(B_{X}F)\overline{Y} + (B_{X}F)\overline{X} - 2A(X)\overline{Y} = 0 \iff (b) (B_{X}F)(\overline{Y},Z) - (B_{X}F)(Z,X) - 2A(X)F(Y,Z) = 0$

(2.11) (a) $(B_{X}F)Y + (B_{X}F)X - A(Y)(B_{X}T - (B_{X}F)X) = A(X)(B_{Y}T - (B_{Y}F)Y) = 0 \iff (b) (B_{X}F)(Y,Z) - (B_{X}F)(Z,X) + A(Y)(B_{X}A)(\overline{Y}) - (B_{X}F)(Z,X) + A(X)(B_{X}A)(\overline{Y}) = 0$

An L-contact manifold is called an almost LS-Sasakian manifold, if

(2.12) (a) $(B_{X}F)(Y,Z) + (B_{X}F)(Z,X) + (B_{X}F)(X,Y) - 4A(X)F(Y,Z) + A(Y)F(Z,X) + A(Z)F(X,Y) = 0$

This gives

(b) $(B_{X}F)(\overline{Y},\overline{Z}) + (B_{X}F)(\overline{Z},\overline{X}) + (B_{X}F)(\overline{X},\overline{Y}) = 0$

An L-Contact manifold is called a generalized L-Co-symplectic manifold, if

(2.13) (a) $(B_{X}F)(Y,Z) + A(Y)(B_{X}A)(\overline{Z}) - A(Z)(B_{X}A)(\overline{Y}) = 0$

This gives

(b) $(B_{X}F)(\overline{Y},\overline{Z}) = 0$

An L-Contact manifold is called a generalised nearly L-Co-symplectic manifold, if
\[ (2.14) \quad (B_F^Y F)(Y, Z) + A(Y)(B_X A)(\overline{Z}) - A(Z)(B_X A)(\overline{Y}) = (B_F^Y F)(Z, X) + A(Z)(B_Y A)(\overline{X}) - A(X)(B_Y A)(\overline{Z}) = (B_Z F)(X, Y) + A(X)(B_Z A)(\overline{Y}) - A(Y)(B_Z A)(\overline{X}) \]

This gives
\[ (b) \quad (B_F^Y F)(\overline{Y}, \overline{Z}) = (B_F^Y F)(\overline{Z}, \overline{X}) = (B_F^Y F)(\overline{X}, \overline{Y}) \]

An L-Contact manifold is a generalized almost L-Co-symplectic manifold, if
\[ (2.15) \quad (B_F^Y F)(Y, Z) + (B_F^Y F)(Z, X) + (B_Z F)(X, Y) - A(X)(B_Y A)(\overline{Z}) - (B_Z A)(\overline{Y}) \]
\[ - A(Y)(B_X A)(\overline{X}) - A(Z)(B_Y A)(\overline{Y}) - (B_Y A)(\overline{X}) = 0 \]
Which implies
\[ (b) \quad (B_F^Y F)(\overline{Y}, \overline{Z}) + (B_F^Y F)(\overline{Z}, \overline{X}) + (B_Z F)(\overline{X}, \overline{Y}) = 0 \]

**PROPERTIES**

From (2.5) (a), we see that in an LS – Sasakian manifold, \( B_T F = 0 \). We will now consider nearly LS-Sasakian manifold

Putting \( T \) for \( X \) in (2.7), we get
\[ (3.1) \quad (B_F^Y F)(Y, Z) = - (B_F^Y A)(\overline{Z}) + 2 F(Y, Z) = (B_Z A)(\overline{Y}) + 2 F(Y, Z) \]

Hence
\[ (3.2) \quad (B_F^Y A)(\overline{Z}) + (B_Z A)(\overline{Y}) = 0 \iff (b) \quad B_T T = 0 \]

Barring \( Y \) and \( Z \) in equation (3.1) and then using (2.4) and (3.2), we get
\[ (3.3) \quad (B_F^Y F)(Y, Z) = -(B_F^Y A)(Z) - 2 F(Y, Z) = (B_Z A)(Y) - 2 F(Y, Z) \]

From (3.1) and (3.3), we obtain
\[ (3.4) \quad (B_F^Y A)(Z) + (B_Z A)(\overline{Y}) = -4 F(Y, Z) \quad (b) \quad (B_F^Y A)(Z) + (B_Z A)(Y) = -4 g(\overline{Y}, \overline{Z}) \]

Hence, on a nearly LS-Sasakian manifold, (3.1), (3.2), (3.3) and (3.4) hold.

Almost LS-Sasakian manifold will now be considered. Putting \( T \) for \( X \) in (2.12) (a), we get
\[ (3.5) \quad (B_F^Y F)(Y, Z) = (B_F^Y A)(\overline{Z}) - (B_Z A)(\overline{Y}) - 4 F(Y, Z) \iff (b) \quad B_T T = 0 \]

Barring \( Y \) and \( Z \) in equation (3.5) (a) and using (2.4), we get
\[ (3.6) \quad (B_F^Y F)(Y, Z) = (B_F^Y A)(Z) - (B_Z A)(Y) + 4 F(Y, Z) \]

From (3.5) (a) and (3.6), we obtain
\[ (3.7) \quad (B_F^Y A)(Z) - (B_F^Y A)(\overline{Z}) - (B_Z A)(Y) + (B_Z A)(\overline{Y}) + 4 F(Y, Z) = 0 \iff \]
\[ (b) \quad (B_F^Y A)(\overline{Z}) + (B_Z A)(\overline{Y}) + (B_F^Y A)(Z) + (B_Z A)(Y) + 8 g(\overline{Y}, \overline{Z}) = 0 \]

from (2.7) and (2.14) (a), we see that

A nearly LS-Sasakian manifold is a generalized nearly L-Co-symplectic manifold, if
\[ (3.8) \quad (B_F^Y A)(\overline{Z}) = 2 F(\overline{X}, \overline{Y}) \iff (b) \quad (B_F^Y A)(Y) = -2 g(\overline{X}, \overline{Y}) \iff (c) \quad B_T T = 2 \overline{X} \]

Also. Making the use of (2.12) (a) and (2.15) (a), we see that

A generalized almost L-Co-symplectic manifold is an almost LS-Sasakian manifold, if
\[ (3.9) \quad (B_F^Y A)(\overline{Z}) - (B_Y A)(\overline{X}) = 4 F(Y, X) \]

**NIJENHUIS TENSOR**

In an L-Contact manifold with the semi-symmetric metric connection \( B \), Nijenhuis tensor is given by
\[ (4.1) \quad \mathcal{N}(X, Y, Z) = (B_F^X F)(Y, Z) + (B_F^Y F)(Z, X) + (B_F^Z F)(Y, \overline{Z}) + (B_Y A)(\overline{X}) \]

Where
\[ \mathcal{N}(X, Y, Z) \equiv g(N(X, Y, Z)) \]

Barring \( X, Y, Z \) in (4.1) and using equations (2.7), we see that a nearly LS-Sasakian manifold is completely integrable, if
\[ (4.2) \quad (B_F^X F)(\overline{Y}, \overline{Z}) + (B_F^Y F)(\overline{Z}, \overline{X}) = 0. \]
Barring X, Y, Z in (4.1) and using equations (2.12) (b), we can prove that an almost LS-Sasakian manifold is completely integrable, if

\[(4.3) \quad (B_{\overrightarrow{Z}} F)(\overrightarrow{X}, \overrightarrow{Y}) = 0.\]

**INDUCED CONNECTION IN AN LS-SASAKIAN MANIFOLD**

Let \(M_{2m-1}\) be submanifold of \(M_{2m+1}\) and let \(c : M_{2m-1} \rightarrow M_{2m+1}\) be the inclusion map such that

\[d \in M_{2m-1} \rightarrow cd \in M_{2m+1},\]

Where \(c\) induces a linear transformation (Jacobian map) \(J : T'_{2m-1} \rightarrow T'_{2m+1}\).

\(T'_{2m-1}\) is a tangent space to \(M_{2m-1}\) at point \(d\) and \(T'_{2m+1}\) is a tangent space to \(M_{2m+1}\) at point \(cd\) such that \(\overrightarrow{X}\) in \(M_{2m-1}\) at \(d\) \(\rightarrow J\overrightarrow{X}\) in \(M_{2m+1}\) at \(cd\)

Let \(\overrightarrow{g}\) be the induced Lorentzian metric in \(M_{2m-1}\). Then we have

\[(5.1) \quad \overrightarrow{g}(\overrightarrow{X}, \overrightarrow{Y}) = \langle (\overrightarrow{g}(\overrightarrow{X}, \overrightarrow{Y})) \rangle b\]

We now suppose that a semi-symmetric metric connection \(B\) in an LS-Sasakian manifold is given by

\[(5.2) \quad B_{\overrightarrow{X}} Y = D_{\overrightarrow{X}} Y + A(Y)X - g(Y, Y)T,\]

Where \(X\) and \(Y\) are arbitrary vector fields of \(M_{2m+1}\). If

\[(5.3) \quad T_1 = j t_1 - \rho_1 M - \sigma_1 N\]

Where \(t_1\) is \(C^n\) vector fields in \(M_{2m-1}\) and \(M\) and \(N\) are unit normal vectors to \(M_{2m-1}\).

Denoting by \(\overrightarrow{D}\) the connection induced on the submanifold from \(D\). Let

\[(5.4) \quad D_{\overrightarrow{JX}} \overrightarrow{Y} = J(\overrightarrow{D_X Y}) - h(\overrightarrow{X}, \overrightarrow{Y}) M - k(\overrightarrow{X}, \overrightarrow{Y}) N\]

Where \(h\) and \(k\) are symmetric bilinear functions in \(M_{2m-1}\). Similarly we have

\[(5.5) \quad B_{\overrightarrow{JX}} \overrightarrow{Y} = J(\overrightarrow{B_X Y}) - p(\overrightarrow{X}, \overrightarrow{Y}) M - q(\overrightarrow{X}, \overrightarrow{Y}) N,\]

Where \(\overrightarrow{B}\) is the connection induced on the submanifold from \(B\) and \(p, q\) are symmetric bilinear functions in \(M_{2m-1}\)

In consequence of (5.2), we have

\[(5.6) \quad B_{\overrightarrow{JX}} \overrightarrow{Y} = D_{\overrightarrow{JX}} \overrightarrow{Y} + A(\overrightarrow{Y}) \overrightarrow{X} - g(\overrightarrow{X}, \overrightarrow{Y}) T_1\]

Using (5.4), (5.5) and (5.6), we get

\[(5.7) \quad J(\overrightarrow{B_X Y}) - p(\overrightarrow{X}, \overrightarrow{Y}) M - q(\overrightarrow{X}, \overrightarrow{Y}) N = J(\overrightarrow{D_X Y}) - h(\overrightarrow{X}, \overrightarrow{Y}) M - k(\overrightarrow{X}, \overrightarrow{Y}) N + A_1(\overrightarrow{Y}) \overrightarrow{X} - g(\overrightarrow{X}, \overrightarrow{Y}) T_1\]

Using (5.3), we obtain

\[(5.8) \quad J(\overrightarrow{B_X Y}) - p(\overrightarrow{X}, \overrightarrow{Y}) M - q(\overrightarrow{X}, \overrightarrow{Y}) N = J(\overrightarrow{D_X Y}) - h(\overrightarrow{X}, \overrightarrow{Y}) M - k(\overrightarrow{X}, \overrightarrow{Y}) N + a_1(\overrightarrow{Y}) \overrightarrow{X} - \overrightarrow{g}(\overrightarrow{X}, \overrightarrow{Y}) (\overrightarrow{Jt_1} - \rho_1 M - \sigma_1 N)\]

Where \(\overrightarrow{g}(\overrightarrow{Y}, t_1) \equiv a_1(\overrightarrow{Y})\)

This implies

\[(5.9) \quad \overrightarrow{B_X Y} = \overrightarrow{D_X Y} + a_1(\overrightarrow{Y}) \overrightarrow{X} - \overrightarrow{g}(\overrightarrow{X}, \overrightarrow{Y}) t_1\]

Iff

\[(5.10) \quad \overrightarrow{g}(\overrightarrow{X}, \overrightarrow{Y}) = \frac{1}{\rho_1} \{ h(\overrightarrow{X}, \overrightarrow{Y}) - p(\overrightarrow{X}, \overrightarrow{Y}) \} = \frac{1}{\sigma_1} \{ k(\overrightarrow{X}, \overrightarrow{Y}) - q(\overrightarrow{X}, \overrightarrow{Y}) \}\]

Therefore,

**Theorem 5.1** The connection induced on a submanifold of an LS-Sasakian manifold with a semi-symmetric metric connection with respect to unit normal vectors \(M\) and \(N\) is also semi-symmetric metric connection iff (5.10) holds.
REFERENCES