In the present paper we prove common fixed point theorems in complete cone metric spaces for weak contraction which generalize and extend some well-known previous results of [9].

**KEYWORDS**: Cone metric spaces, Common fixed point Picard iteration, Weak Contraction.

**INTRODUCTION**

Fixed point theory as an important branch of nonlinear functional analysis theory has been applied in many disciplines, see for instance [6-8]. By replacing the real numbers with an ordered Banach space, Huang and Zhang [1] defined cone metric spaces and proved some fixed point theorems of contractions on cone metric spaces. Since several authors have studied the fixed point problem of nonlinear mappings in cone metric spaces see for instance [1],[2],[3],[4],[5].

The purpose of this paper is to extend and prove some common fixed point of general contractions in cone metric spaces. Our results generalize and the respective theorems of [9].

**PRELIMINARY NOTES**

First, we recall some standard notations and definitions in cone metric spaces properties [1].

**Definition 2.1** [1]: Let $E$ be a real Banach space and $P$ be a subset of $E$. $P$ is called a cone if and only if:

(i) $P$ is closed, nonempty and $P \neq \{0\}$,

(ii) $ax + by \in P$ for all $x, y \in P$ and nonnegative real number $a, b$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \iff P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on $E$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x \preceq y$ if $y - x \in \text{int } P$. $\text{int } P$ denotes the interior of $P$. Note that, it is clear if $a \leq b, c \leq d$, then $a + c \leq b + d$, and for $\mu \in R, \mu \geq 0, a\mu \leq b\mu$.

The cone $P$ is called normal if there is a number $K > 0$ such that $x, y \in E, 0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$.

In following we always suppose $E$ is a Banach space. $P$ is a cone in $E$ with $\text{int } P \neq \emptyset$ and is partial ordering with respect

**Definition 2.2** [1]: Let $X$ be a nonempty set. Suppose the mapping $d: X \times X \rightarrow E$ satisfies

(i) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, x) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space.

**Definition 2.3** [9]: Let $(X, d)$ be a cone metric space. $T$ a self map of $X$. Let $x_{2n+1} = f(T, x_{2n})$ be some iteration procedure. Suppose that $T^p(T)$, the fixed point set of, is nonempty and that $x_{2n}$ to a point $p \in T(T)$. Let $\{y_{2n}\} \subset X$, and define $\epsilon_{2n} = d(y_{2n+1}, f(T, y_{2n}))$. If $\lim_{n \rightarrow \infty} \epsilon_{2n} = 0$ implies $\lim_{n \rightarrow \infty} y_{2n} = p$, then $x_{2n+1} = f(T, x_{2n})$ is called stable with respect $T$. 

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**ABSTRACT**

In the present paper we prove common fixed point theorems in complete cone metric spaces for weak contraction which generalize and extend some well-known previous results of [9].

**KEYWORDS**: Cone metric spaces, Common fixed point Picard iteration, Weak Contraction.
MAIN RESULTS

The following theorem is an extension of theorem 2.1 and 3.2 of [9].

**Theorem 3.1** Let $X$ be a nonempty complete cone metric space. $T_1, T_2 : X \times X \to X$ be any two self maps on $X$ such that

$$d(T_1x, T_2y) \leq h(x, y) + Ld(y, T_1x)$$

(3.1.1)

For all $x, y \in X$, where $h$ is some real number in $[0, 1]$, and $L$ is some real number in $(0, \infty)$. Then $T_1$ and $T_2$ have a common fixed point in $X$.

**Proof.** Let $x_0 \in X$ and $n \geq 1$, Let $\{x_{2n}\}$ be a sequence generated by the following Picard iteration

$$x_{2n} = T_1 x_{2n-1} \text{ and } x_{2n+1} = T_2 x_{2n}$$

Now Taking $x = x_{2n-1}$ and $y = x_{2n}$. Then we get

$$d(T_1 x_{2n-1}, T_2 x_{2n}) \leq hd(x_{2n-1}, x_{2n}).$$

This implies that

$$d(x_{2n}, x_{2n+1}) \leq hd(x_{2n-1}, x_{2n}).$$

In general

$$d(x_{2n}, x_{2n+1}) \leq ... \leq hd(x_{2n-1}, x_{2n}) \leq ... \leq shd(x_1, x_0)$$

For $n \geq m$, we have

$$d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n-2}) + ... + d(x_{2m}, x_{2m})$$

$$\leq (h^{2n-1} + h^{2n-2} + ... + h^{2m})d(x_1, x_0)$$

$$= \frac{h^{2m}}{1-h}d(x_1, x_0)$$

For a given $c \in E$ with $0 \leq c$, that is, $c \in \text{int}P$, there exist $B(0, r)$ such that $c+ B(0, r) \subseteq P$, where $B(0, r) = \{x \in E, \|x\| \leq r\}$. But there exist a positive number $N$ such that $\frac{h^m}{1-h}d(x_1, x_0) \in B(0, r)$ for all $m > N$. Therefore, we have

$$d(x_{2n}, x_{2m}) \leq \frac{h^{2m}}{1-h}d(x_1, x_0) \leq c, \text{for all } m > N.$$ This implies that $\{x_{2n}\}$ is a Cauchy sequence and is convergent because of the completeness of $X$. We denote $p = \lim_{n \to \infty} x_{2n}$. Notice that

$$d(p, T_1p) \leq d(p, x_{2n+1}) + d(x_{2n+1}, T_1p)$$

Lower bound

$$d(p, T_1p) \leq d(p, x_{2n+1}) + d(T_1x_{2n}, T_1p)$$

$$\leq d(p, x_{2n+1}) + hd(x_{2n}, x_{2n}) + Ld(p, T_1x_{2n})$$

$$= (1 + L) d(p, x_{2n+1}) + hd(x_{2n}, x_{2n}) + Ld(p, T_1x_{2n})$$

$$= (1 + L) d(p, x_{2n+1}) + hd(x_{2n}, x_{2n}) + Ld(p, T_1x_{2n})$$

This implies that $\lim (p, T_1p) = 0$. Hence $T_1p = p$. Therefore $p$ is a fixed point of $T_1$.

Now $q$ is another fixed point of $T_2$. Then we have

$$d(p, q) \leq d(p, q) \text{ implies that } p = q.$$

Therefore, the fixed point of $T_1$ is unique. Similarly, it can be established that $T_2p = p$. Hence $T_1p = p = T_2p$. Thus $p$ is the common fixed point of $T_1$ and $T_2$.

**Theorem 3.2** Let $X$ be a nonempty complete cone metric space. $T_1, T_2 : X \times X \to X$ be any two self - maps on $X$ such that

$$d(T_1x, T_2y) \leq h(x, y) + Ld(x, T_2y)$$

(3.2.1)

For all $x, y \in X$, where $h$ is some real number in $[0, 1]$, and $L$ is some real number in $(0, \infty)$. Then $T_1$ and $T_2$ have a common fixed point in $X$.

**Proof:** The proof of this theorem is similar to the proof of the theorem 3.1.

**REFERENCES**


