ON THE TEMPERATURE IN A NON-HOMOGENEOUS BAR IN ASSOCIATION WITH $H$–FUNCTION

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ABSTRACT

We employ the $H$ -function to obtain the formal solution of the partial differential equation:

$$\frac{\partial v}{\partial t} = \lambda \frac{\partial}{\partial u} \left[ (1-u^2) \frac{\partial v}{\partial u} \right]$$

Related to a problem of heat conduction by making use of the integral and orthogonality property of Jacobi polynomials. The result generalizes a number of known particular cases on specialization of the parameters.

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INTRODUCTION

The $H$ -function occurring in the paper will be defined and represented by Buschman and Srivastava [1] as follows:

$$H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N}[z] \begin{cases} (a_j, \alpha_j; A_j h_{N,X} (a_j, \alpha_j)h_{N,Y}, P, R) \\ (b_j, \beta_j h_{M,X} (b_j, \beta_j)h_{M,Y}, Q) \end{cases}$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \phi(\xi) z^\xi d\xi$$

where

$$\phi(\xi) = \prod_{j=1}^{M} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{N} \{\Gamma(1 - a_j + \alpha_j \xi)\}^A_j$$

$$\prod_{j=M+1}^{Q} \{\Gamma(1 - b_j + \beta_j \xi)\}^B_j \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j \xi)$$

(1.2)

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j (j = 1, \ldots, P)$ and $b_j (j = 1, \ldots, Q)$ are complex parameters, $\alpha_j \geq 0 (j = 1, \ldots, P), \beta_j \geq 0 (j = 1, \ldots, Q)$ (not all zero simultaneously) and exponents $A_j (j = 1, \ldots, N)$ and $B_j (j = N + 1, \ldots, Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $H$ -function given by equation (1.1) have been given by (Buschman and Srivastava[1]).

$$\Omega \equiv \sum_{j=1}^{M} |\beta_j| + \sum_{j=1}^{N} |A_j| - \sum_{j=M+1}^{P} |\beta_j| - \sum_{j=N+1}^{Q} |\alpha_j| > 0$$

(1.3)
and \(|\arg(z)| < \frac{1}{2} \pi \Omega\) \hspace{1cm} (1.4)

The behavior of the \(H\) function for small values of \(|z|\) follows easily from a result recently given by (Rathie [5], p.306, eq.(6.9)).

We have

\[
\overline{H}^{M,N}_{P,Q}(z) = 0 \left(\frac{1}{z}\right), \gamma = \min_{1 \leq j \leq N} \left[ \text{Re} \left( \frac{b_j}{\beta_j} \right) \right], |z| \to 0
\] \hspace{1cm} (1.5)

If we take \(A_j = 1(\ j = 1, \ldots, N), \ B_j = 1(\ j = M + 1, \ldots, Q)\) in (1.1), the function \(\overline{H}^{M,N}_{P,Q}\) reduces to the Fox’s \(H\)-function [3].

We shall use the following notation:

\[
A^* = (a_j, \alpha_j; A_j)_{N+1, p} \quad \text{and} \quad B^* = (b_j, \beta_j; B_j)_{M+1, Q}
\]

We require the following result:

\[
(1-u)^{r}(1+u)^{s} P_{r}^{(\alpha, \beta)}(u) P_{s}^{(\alpha, \beta)}(u) = \frac{\alpha r \Gamma(\alpha + r + 1) \Gamma(\alpha + r + 1)}{\beta s \Gamma(\beta + s + 1) \Gamma(\beta + s + 1)} \sum_{c=0}^{r} (A_c; c) d^c (-1)^c 2^{\alpha r+1} \Gamma(1+\beta+r) \Gamma(1+\alpha+r)
\] \hspace{1cm} (1.6)

Where \(\text{Re}(\alpha) > -1, \text{Re}(\beta) > -1\) and \(\Omega > 0, |\arg z| < \frac{1}{2} \Omega \pi, h > 0, P \leq Q(Q+1) \text{ and } |d| < 1, \ g > 0\).

This orthogonality property of the Jacobi polynomials:

\[
\int_{-1}^{1} (1-u)^{r}(1+u)^{s} P_{r}^{(\alpha, \beta)}(u) P_{s}^{(\alpha, \beta)}(u) du = h_{r,s} \delta_{rs}
\] \hspace{1cm} (1.7)

Where

\[
\text{Re}(\alpha) > -1, \text{Re}(\beta) > -1 \text{ and } \quad h_{r,s} = \frac{2^{\alpha+\beta+1} \Gamma(\alpha + r + 1) \Gamma(\beta + r + 1)}{r! \Gamma(\alpha + r + 1 + 2r) \Gamma(\beta + r + 1 + r)}
\]

And \(\delta_{rs}\) is Kronecker delta function, defined as:

\[
\delta_{rs} = \begin{cases} 1, & \text{if } r = s \cr 0, & \text{if } r \neq s \end{cases}
\]

**HEAT CONDUCTION AND \(H\)-FUNCTION**

Our problem is to find a function \(v(u, t)\) representing the tempreature in a non-homogeneous bar with ends at \(u = \pm 1\) in which the thermal conductivity is proportional to \((1-u^2)\) and if the ateral surface of the bar is insulated, it satisfies the partial differential equation of heat conduction.
\[
\frac{\partial v}{\partial t} = \lambda \frac{\partial}{\partial u} \left( 1 - u^2 \right) \frac{\partial v}{\partial u} \quad (2.1)
\]

Where \( \lambda \) is a constant, provided the thermal coefficient is constant.

The boundary conditions of the problem are that both ends of a bar at \( u = \pm 1 \) are also insulated because the conductivity vanishes there, and the initial conditions:

\[
v(u,0) = f(u), \quad -1 < u < 1 \quad (2.2)
\]

In view of (2.2), we consider

\[
v = f(u) = (1-u)^\alpha P_{Q}^{(\alpha, \beta)}(u)
\]

Where \( P_{Q}^{(\alpha, \beta)}(u) \) is a Jacobi polynomial.

Equation (2.3) is valid since \( f(u) \) is continuous in the closed interval \(-1 \leq u \leq 1\) and has a piece-wise continuous derivative there, then \( \alpha > -1, \beta > -1 \), the Jacobi series associated with \( f(u) \) converges uniformly to \( f(u) \) in \(-1+ \leq u \leq 1- e, 0 < e < 1 \).

Now multiplying both sides of (2.5) by \( (1-u)\alpha (1+u)\beta P_{Q}^{(\alpha, \beta)}(u); \alpha > -1, \beta > -1 \) and integrating from \(-1 \) to \( 1 \), the use of (1.6) yields

\[
A_\nu = \frac{1}{h} \left( 1 - u \right) e^{u \alpha} (1 + u)^\beta P_{Q}^{(\alpha, \beta)}(u) \left[ \frac{\alpha}{\nu} ; \frac{\Gamma(1+\alpha+\beta+r)}{\Gamma(1+\alpha+r)} \right]
\]

(2.6)

We freely apply (1.6) in (2.6) to obtain

\[
A_\nu = \left( -1 \right)^{2 \alpha + \beta + 1} \frac{\Gamma(1+\alpha + \beta + r) \Gamma(1+\alpha+\beta+2r)}{\Gamma(1+\alpha+r)}
\]

(2.7)

On substituting the value of \( A_\nu \) from (2.7) in (2.4), we arrive at the desired solution.

\[
v(u,t) = 2^\nu \sum_{w,v=0} \left[ \frac{\alpha}{\nu} ; \frac{\Gamma(1+\alpha+\beta+2r)}{\Gamma(1+\alpha+r)} \right] e^{u \alpha} (1 + u)^\beta P_{Q}^{(\alpha, \beta)}(u) e^{x \alpha}
\]

(2.8)
Where
\[ f(w) = \frac{(-1)^w 2^\alpha \beta \gamma (1 + \alpha + \beta + w) \Gamma(1 + \alpha + \beta + 2w)}{\Gamma(1 + \alpha + w)} \frac{(A_\nu; c) d^\nu}{(B_\nu; c) e^!} \]

And the conditions of validity are:
\[ \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \Omega > 0, |\arg z| < \frac{1}{2} \Omega \pi, h > 0, P \leq Q(Q + 1 \text{ and } d |< 1), g > 0. \]

**SPECIAL CASE**
If we take \( A_j = 1 \), \( B_j = 1 \), we get the result due to Chaurasia [3].

**REFERENCES**