ABSTRACT

The aim of this paper is to correct the bounds for the zeros of certain polynomials recently proved by K.A.Kareem and A.A.Mogbademu [3].

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KEYWORDS: Coefficient, Open Disc, Polynomial, Zeros

I. INTRODUCTION

In the framework of Enestrom-Kakeya theorem([4],[5]) which states that “all the zeros of a polynomial

$$P(z) = \sum_{j=0}^{n} a_j z^j$$ with $$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0$$ lie in $$|z| \leq 1$$”, K.A.Kareem and A.A.Mogbademu [3] recently, while generalizing some results of Gardner and Shields [1] on polynomials with certain monotonicity conditions on their coefficients, claim to have proved the following results:

**Theorem A:** Let $$P(z) = \sum_{j=0}^{n} a_j z^j$$ be a polynomial of degree $$n$$ where for some $$R>0$$,

$$0 < \rho |a_0| \leq R |a_1| \leq R^2 |a_2| \leq \ldots \leq R^{k-1} |a_{k-1}| \leq R^k |a_k| \geq R^{k+1} |a_{k+1}| \geq \ldots \geq R^{n-1} |a_{n-1}| \geq (R - \mu) R^{n-2} |a_n|,$$

and $$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$$ for $$1 \leq j \leq n$$ and for some real $$\alpha$$ and $$\beta$$. Then for $$0 < \delta < 1$$ the number of zeros of $$P(z)$$ in $$|z| \leq \delta R$$ is less than

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0|}$$

where

$$M = |a_0| R \left[ \frac{1}{\rho} + \frac{\mu}{|a_0|} - \cos \alpha - \sin \alpha \right] + 2 |a_k| R^{k+1} \cos \alpha +$$

$$+ |a_n| R^{n+1} \left[ 1 + \frac{\mu}{|a_n|} - \cos \alpha + \sin \alpha \right] + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| R^{j+1}.$$

**Theorem B:** Let $$P(z) = \sum_{j=0}^{n} a_j z^j$$ be a polynomial of degree $$n$$ where $$\text{Re}(a_j) = \alpha_j$$ and $$\text{Im}(a_j) = \beta_j$$ for $$0 \leq j \leq n$$. Suppose that for some $$R>0$$, $$0 \leq \mu < 1$$, $$0 < \rho \leq 1$$, $$0 \leq k \leq n$$,
0 ≠ ρα₀ ≤ Rα₁ ≤ R²α₂ ≤ ......... ≤ R⁴α₄ ≤ Rᵏαₖ
≥ Rᵏ⁺₁αₖ₊₁ ≥ ......... ≥ Rⁿ⁻¹αₙ₊₁ ≥ (R − μ)Rⁿ⁻¹αₙ.

Then for 0 < δ < 1 the number of zeros of P(z) in |z| ≤ δR is less than

$$\frac{1}{\log \frac{1}{δ}} \log |α₀|$$

where

$$M = \frac{1}{ρ} |α₀ − μ| − α₀)R + (|α_n − μ| − α_n)Rⁿ⁺¹ + μR(1 + Rⁿ) + 2α₂Rᵏ⁺¹ + 2\sum_{j=0}^{n} |β_j| R^{j+1}.$$

Theorem C: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where

Re$(a_j) = α_j$ and Im$(a_j) = β_j$ for 0 ≤ j ≤ n. Suppose that for some R > 0,

0 ≤ ρ ≤ 1,0 < λ < 1,0 < ρ₁ ≤ 1,0 < ρ₂ ≤ 1 and for some 0 ≤ k ≤ n,0 ≤ l ≤ n,

0 ≠ ρ₁α₀ ≤ Rα₁ ≤ R²α₂ ≤ ......... ≤ Rᵏαₖ ≤ Rᵏ⁺₁αₖ₊₁ ≥ ......... ≥ Rⁿ⁻¹αₙ₊₁ ≥ (R − μ)Rⁿ⁻¹αₙ

and

ρ₃β₀ ≤ Rβ₁ ≤ R²β₂ ≤ ......... ≤ Rᵏβₖ ≤ Rᵏ⁺₁βₖ₊₁ ≥ ......... ≥ Rⁿ⁻¹βₙ₊₁ ≥ (R − μ)Rⁿ⁻¹βₙ.

Then for 0 < δ < 1 the number of zeros of P(z) in the disc |z| ≤ δR is less than

$$\frac{1}{\log \frac{1}{δ}} \log |α₀|$$

where

$$M = \frac{1}{ρ₁} |α₀ − μ| − α₀)R + (|β₀ − λ| − β₀)R + (μ + λ)R(1 + Rⁿ) + 2α₂Rᵏ⁺¹ + (|α_n − μ| − α_n)Rⁿ⁺¹ + (|β_n − λ| − β_n)Rⁿ⁺¹ + 2β₁R^{l+1}.$$

II. MAIN RESULTS

Unfortunately, there are various mistakes in the proofs of these theorems and the bounds obtained for the moduli of the zeros are not correct. In this paper we give the correct bounds for the zeros of the polynomials in the above mentioned theorems and prove

Theorem 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where for some R > 0,

0 ≤ μ < 1,0 < ρ ≤ 1,0 ≤ k ≤ n,

0 < ρ|aₖ| ≤ R|a₁| ≤ R²|a₂| ≤ ......... ≤ Rᵏ⁺₁|aₖ₊₁| ≤ Rᵏ|aₖ| ≥ Rᵏ⁺₁|aₖ₊₁| ≥ ......... ≥ Rⁿ⁻¹|aₙ₊₁| ≥ (R − μ)Rⁿ⁻¹|aₙ|,
and \( |\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2} \) for \( 1 \leq j \leq n \) and for some real \( \alpha \) and \( \beta \). Then for \( 0 < \delta < 1 \) the number of zeros of \( P(z) \) in \( |z| \leq \delta R \) is less than

\[
\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0|}
\]

where

\[
M = 2|a_0| - \rho|a_0|R(1 - \cos \alpha - \sin \alpha) + 2|a_k|R^{k+1} \cos \alpha \\
+ |a_n|R^{n+1}(1 - \cos \alpha + \sin \alpha) + \mu|a_n|R^{n}(1 + \cos \alpha - \sin \alpha) \\
+ 2 \sin \alpha \sum_{j=0}^{n-1} |a_j|R^{j+1}.
\]

Taking \( R=1 \) in Theorem 1, we get the following

**Corollary 1:** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) where for some
\[
0 \leq \mu < 1, 0 < \rho \leq 1, 0 \leq k \leq n,
\]

\[
0 < \rho |a_0| \leq |a_1| \leq |a_2| \leq \ldots \leq |a_{k-1}| \leq |a_k| \geq |a_{k+1}| \geq \ldots \geq |a_{n+1}| \geq |1 - \mu|a_n|,
\]

and \( |\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2} \) for \( 1 \leq j \leq n \) and for some real \( \alpha \) and \( \beta \). Then for \( 0 < \delta < 1 \) the number of zeros of \( P(z) \) in \( |z| \leq \delta R \) is less than

\[
\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0|}
\]

where

\[
M = 2|a_0| - \rho|a_0|(1 - \cos \alpha - \sin \alpha) + 2|a_k|\cos \alpha \\
+ |a_n|(1 - \cos \alpha + \sin \alpha) + \mu|a_n|(1 + \cos \alpha - \sin \alpha) \\
+ 2 \sin \alpha \sum_{j=0}^{n-1} |a_j|.
\]

For \( \mu = 0, \rho = 1 \), we get the following result of Gardner and Sheilds [1] from Theorem 1.

**Corollary 2:** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) where for some \( R>0 \),
\[
0 \leq k \leq n,
\]

\[
0 < |a_0| \leq R|a_1| \leq R^2|a_2| \leq \ldots \leq R^{k-1}|a_{k-1}| \leq R^k|a_k| \geq R^{k+1}|a_{k+1}| \geq \ldots \geq R^{n-1}|a_{n+1}| \geq R^n|a_n|,
\]

and \( |\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2} \) for \( 1 \leq j \leq n \) and for some real \( \alpha \) and \( \beta \). Then for \( 0 < \delta < 1 \) the number of zeros of \( P(z) \) in \( |z| \leq \delta R \) is less than
where
\[ M = |a_0| R(1 - \cos \alpha - \sin \alpha) + 2|a_k| R^{k+1} \cos \alpha \]
\[ + |a_n| R^{n+1} (1 - \cos \alpha + \sin \alpha) + 2 \sin^2 \alpha \sum_{j=0}^{n-1} |a_j| R^{j+1}. \]

**Theorem 2**: Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) where \( \text{Re}(a_j) = \alpha_j \) and \( \text{Im}(a_j) = \beta_j \) for \( 0 \leq j \leq n \). Suppose that for some \( R > 0 \), \( 0 \leq \mu < 1, 0 < \rho \leq 1, 0 \leq k \leq n \),
\[ 0 \neq \rho \alpha_0 \leq R \alpha_1 \leq R^2 \alpha_2 \leq \ldots \leq R^{k-1} \alpha_{k-1} \leq R^k \alpha_k \]
\[ \geq R^{k+1} \alpha_{k+1} \geq \ldots \geq R^{n+1} \alpha_{n+1} \geq (R - \mu) R^{n+1} \alpha_n. \]
Then for \( 0 < \delta < 1 \) the number of zeros of \( P(z) \) in \( |z| \leq \delta R \) is less than
\[ \frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0|} \]
where
\[ M = 2|a_0| R - \rho (\alpha_0 + |\alpha_0|) R + 2 \alpha_k R^{k+1} + (|\alpha_n| - \alpha_n) R^{n+1} + \mu (\alpha_n + |\alpha_n|) R^n \]
\[ + 2 \sum_{j=0}^{n} |\beta_j| R^{j+1}. \]

Taking \( R = 1 \) in Theorem 2, we get the following

**Corollary 3**: Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) where \( \text{Re}(a_j) = \alpha_j \) and \( \text{Im}(a_j) = \beta_j \) for \( 0 \leq j \leq n \). Suppose that for some \( 0 \leq \mu < 1, 0 < \rho \leq 1, 0 \leq k \leq n \),
\[ 0 \neq \rho \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{k-1} \leq \alpha_k \]
\[ \geq \alpha_{k+1} \geq \ldots \geq \alpha_{n+1} \geq (1 - \mu) \alpha_n. \]
Then for \( 0 < \delta < 1 \) the number of zeros of \( P(z) \) in \( |z| \leq \delta R \) is less than
\[ \frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0|} \]
where
\[ M = 2|a_0| - \rho (\alpha_0 + |\alpha_0|) + 2 \alpha_k + (|\alpha_n| - \alpha_n) + \mu (\alpha_n + |\alpha_n|) \]
\[ + 2 \sum_{j=0}^{n} |\beta_j| \]

Taking \( \mu = 0, \rho = 1 \), we get the following result of Gardner and Shields [1] from Theorem 2.

**Corollary 4**: Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) where \( \text{Re}(a_j) = \alpha_j \) and \( \text{Im}(a_j) = \beta_j \) for \( 0 \leq j \leq n \). Suppose that for some \( R > 0, 0 \leq k \leq n \),
\[ 0 \neq \alpha_0 \leq R \alpha_1 \leq R^2 \alpha_2 \leq \ldots \leq R^{k-1} \alpha_{k-1} \leq R^k \alpha_k \]
\[ \geq R^{k+1} \alpha_{k+1} \geq \ldots \geq R^{n+1} \alpha_{n+1} \geq (R - \mu) R^{n+1} \alpha_n. \]
Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in $|z| \leq \delta R$ is less than

$$
\frac{1}{\log \frac{M}{|a_0|}} \log \frac{M}{|a_0|}
$$

where

$$M = (\alpha_0 - \alpha_0) R + 2\alpha_k R^{k+1} + (|\alpha_n| - |\alpha_n|) R^{n+1} + 2 \sum_{j=0}^{n} |\beta_j| R^{j+1} .
$$

**Theorem 3:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$ where

$$\text{Re}(a_j) = \alpha_j \quad \text{and} \quad \text{Im}(a_j) = \beta_j \quad \text{for} \quad 0 \leq j \leq n .$$

Suppose that for some $R > 0$,

$0 \leq \mu < 1, 0 \leq \lambda < 0, 0 \leq \rho_1 < \rho_2 < 0$ and for some $0 \leq k \leq n, 0 \leq l \leq n,$

$0 \neq \rho_1 \alpha_0 \leq R \alpha_0 \leq R^2 \alpha_2 \leq \ldots \ldots \leq R^{k+1} \alpha_{k+1} \leq R^k \alpha_k$

$\geq R^k \alpha_{k+1} \geq \ldots \ldots \geq R^n \alpha_{n+1} \geq (R - \mu) R^{n+1} \alpha_n$

and

$\rho_2 \beta_0 \leq R \beta_1 \leq R^2 \beta_2 \leq \ldots \ldots \leq R^{l+1} \beta_{l+1} \leq R \beta_l$

$\geq R^{l+1} \beta_{l+1} \geq \ldots \ldots \geq R^n \beta_{n+1} \geq (R - \lambda) R^{n+1} \beta_n .
$

Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disc $|z| \leq \delta R$ is less than

$$
\frac{1}{\log \frac{M}{|a_0|}} \log \frac{M}{|a_0|}
$$

where

$$M = 2|\alpha_0| R - \rho_2 R (|\alpha_0| + |\alpha_0|) + 2 \alpha_k R^{k+1} + (|\alpha_n| - |\alpha_n|) R^{n+1} + \mu (\alpha_n + |\alpha_n|) R^n$

$$+ 2 |\beta_0| R - \rho_2 R (|\beta_0| + |\beta_0|) + 2 \beta_l R^{l+1} + (|\beta_n| - |\beta_n|) R^{n+1} + \lambda (\beta_n + |\beta_n|) R^n .
$$

Taking $\rho_1 = \rho_2 = 1, \lambda = \mu = 0$ in Theorem 3, we get the following result due to Gardner and Sheils [1]:

**Corollary 5:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$ where

$$\text{Re}(a_j) = \alpha_j \quad \text{and} \quad \text{Im}(a_j) = \beta_j \quad \text{for} \quad 0 \leq j \leq n .$$

Suppose that for some $R > 0, 0 < \rho \leq 1$ and for some $0 \leq k \leq n,$

$0 \neq \alpha_0 \leq R \alpha_0 \leq R^2 \alpha_2 \leq \ldots \ldots \leq R^{k+1} \alpha_{k+1} \leq R^k \alpha_k$

$\geq R^{k+1} \alpha_{k+1} \geq \ldots \ldots \geq R^n \alpha_{n+1} \geq R^\alpha \alpha_n$

and

$\beta_0 \leq R \beta_1 \leq R^2 \beta_2 \leq \ldots \ldots \leq R^{l+1} \beta_{l+1} \leq R \beta_l$

$\geq R^{l+1} \beta_{l+1} \geq \ldots \ldots \geq R^n \beta_{n+1} \geq R^\beta \beta_n .
$

Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disc $|z| \leq \delta R$ is less than
For the proof of Theorem 1: Consider the polynomial
\[ F(z) = (R - z)(a_0 + a_1z + a_2z^2 + \ldots + a_{k-1}z^{k-1} + a_kz^k + a_{k+1}z^{k+1} + \ldots + a_{n-1}z^{n-1} + a_nz^n) \]
\[ = a_0R + (a_1R - a_0)z + (a_2R - a_1)z^2 + \ldots + (a_{k-1}R - a_{k-2})z^{k-1} + (a_kR - a_{k-1})z^k + \ldots + (a_{n-1}R - a_{n-2})z^{n-1} + (a_nR - a_{n-1})z^n - a_nz^{n+1}. \]

For \( |z| \leq R \), we have, by using the hypothesis and the Lemma,
\[ |F(z)| \leq |a_0|R + |a_1R - \rho a_0 + \rho a_0 - a_0|R + |a_2R - a_1|R^2 + \ldots + |a_{k-1}R - a_{k-2}|R^{k-1} + |a_kR - a_{k-1}|R^k \]
\[ + |a_{k+1}R - a_k|R^{k+1} + \ldots + |a_{n-1}R - a_{n-2}|R^{n-1} + |a_nR - \mu a_n + \mu a_n - a_{n-1}|R^n + |a_n|R^{n+1} \]
\[ \leq |a_0|R + [(|a_1R - \rho a_0|)\cos \alpha + (|a_1R + \rho a_0|)\sin \alpha]R + (1 - \rho)|a_0|R \]
\[ + [(|a_2R - a_1|)\cos \alpha + (|a_2R + a_1|)\sin \alpha]R^2 + \ldots \]

III. LEMMAS

For the proof of the above results, we make use of the following lemma which is due to Govil and Rahman [2]:

**Lemma 1:** For any two complex numbers \( z_1, z_2 \) such that \( |z_1| \geq |z_2| \) and for some real \( \alpha, \beta \),
\[ |z_1 - z_2| \leq (|z_1| - |z_2|)\cos \alpha + (|z_1| + |z_2|)\sin \alpha. \]

**Lemma 2:** Let \( f(z) \) be analytic, \( |f(z)| \leq M \) for \( |z| \leq R \) and \( f(0) \neq 0 \). Then the number of zeros of \( f(z) \) in \( |z| \leq \delta R \) where \( 0 < \delta < 1 \) is less than or equal to
\[ \frac{1}{\log \frac{1}{\delta}} \log M. \]

For Lemma 2 see [6].

IV. PROOFS OF THEOREMS

**Proof of Theorem 1:** Consider the polynomial
\[ F(z) = (R - z)(a_0 + a_1z + a_2z^2 + \ldots + a_{k-1}z^{k-1} + a_kz^k + a_{k+1}z^{k+1} + \ldots + a_{n-1}z^{n-1} + a_nz^n) \]
\[ = a_0R + (a_1R - a_0)z + (a_2R - a_1)z^2 + \ldots + (a_{k-1}R - a_{k-2})z^{k-1} + (a_kR - a_{k-1})z^k + \ldots + (a_{n-1}R - a_{n-2})z^{n-1} + (a_nR - a_{n-1})z^n - a_nz^{n+1}. \]
\[
+ \left[\left(\left|a_{k-1}\right| R - \left|a_{k-2}\right| \right) \cos \alpha + \left(|a_{k-1}| R + |a_{k-2}| \sin \alpha \right) R^{k-1}\right] \\
+ \left[\left(\left|a_{k}\right| R - \left|a_{k-1}\right| \right) \cos \alpha + \left(|a_{k}| R + |a_{k-1}| \sin \alpha \right) R^k\right] \\
+ \left(|a_{k+1}| - \left|a_{k}\right| R \cos \alpha + \left(|a_{k}| R \sin \alpha \right) R^{k+1}\right] + \ldots \\
+ \left[\left(\left|a_{n-2}\right| - \left|a_{n-1}\right| R \cos \alpha + \left(|a_{n-1}| R \sin \alpha \right) R^{n-1}\right] \\
+ \left[\left(\left|a_{n-1}\right| - (R - \mu) |a_n| \right) \cos \alpha + \left(|a_{n-1}| + (R - \mu) |a_n| \right) \sin \alpha \right] R^n \\
+ \mu |a_n| R^n + |a_n| R^{n+1}
\]

\[
= |a_0| R + (1 - \rho) |a_0| R - \rho |a_0| R \cos \alpha + \rho |a_0| R \sin \alpha + 2|a_k| R^{k+1} \cos \alpha - |a_n| R^{n+1} \cos \alpha \\
+ |a_n| R^{n+1} \sin \alpha + \mu |a_n| R^n \cos \alpha - \mu |a_n| R^n \sin \alpha + \mu |a_n| R^n + |a_n| R^{n+1}
\]

\[
= 2|a_0| R - \rho |a_0| R \cos \alpha + \rho |a_0| R \sin \alpha + 2\rho |a_0| R \sin \alpha + 2|a_k| R^{k+1} \cos \alpha \\
- |a_n| R^{n+1} \cos \alpha + |a_n| R^{n+1} \sin \alpha + \mu |a_n| R^n \cos \alpha - \mu |a_n| R^n \sin \alpha + \mu |a_n| R^n + |a_n| R^{n+1}
\]

Since \(F(z)\) is also analytic for \(|z| \leq \delta R, \delta > 0\), it follows by the lemma that the number of zeros of \(F(z)\) in 

\[|z| \leq \delta R, 0 < \delta < 1\] is less than or equal to 

\[
\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|F(0)|} = \frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0| R}.
\]

Since the zeros of \(P(z)\) are also the zeros of \(F(z)\), it follows that the number of zeros of \(P(z)\) in 

\[|z| \leq \delta R, 0 < \delta < 1\] is less than or equal to 

\[
\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|F(0)|} = \frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0| R}.
\]

That proves theorem 1.
Proof of Theorem 2: Consider the polynomial

\[ F(z) = (R - z)(a_0 + a_1 z + a_2 z^2 + \ldots + a_{k-1} z^{k-1} + a_k z^k + a_{k+1} z^{k+1} + \ldots + a_{n-1} z^{n-1} + a_n z^n) \]

\[ = a_0 R + (a_1 R - a_0) z + (a_2 R - a_1) z^2 + \ldots + (a_{k-1} R - a_{k-2}) z^{k-1} + (a_k R - a_{k-1}) z^k \]

\[ + (a_{k+1} R - a_k) z^{k+1} + \ldots + (a_{n-1} R - a_{n-2}) z^{n-1} + (a_n R - a_{n-1}) z^n - a_n z^{n+1}. \]

\[ = a_0 R + (\alpha_1 R - \alpha_0) z + (\alpha_2 R - \alpha_1) z^2 + \ldots + (\alpha_{k-1} R - \alpha_{k-2}) z^{k-1} + (\alpha_k R - \alpha_{k-1}) z^k \]

\[ + (\alpha_{k+1} R - \alpha_k) z^{k+1} + \ldots + (\alpha_{n-1} R - \alpha_{n-2}) z^{n-1} + (\alpha_n R - \alpha_{n-1}) z^n - a_n z^{n+1}. \]

For \( |z| \leq R \), we have, by using the hypothesis, \[ |F(z)| \leq |a_0| R + |\alpha_1 R - \alpha_0| R + |\alpha_2 R - \alpha_1| R^2 + \ldots + |\alpha_{k-2} R - \alpha_{k-3}| R^{k-1} + |\alpha_k R - \alpha_{k-1}| R^k \]

\[ + |a_{k+1} R - a_k| R^{k+1} + \ldots + |a_{n-1} R - a_{n-2}| R^{n-1} + |\alpha_n R - \alpha_{n-1}| R^n + |a_n R^{n+1}| + |\beta_0| R^{\alpha_0} R^2 + \ldots + |\beta_{n-2} R - \beta_{n-3}| R^{n-1} + |\beta_n R^{n+1}| \]

\[ \leq |a_0| R + |\alpha_1 R - \alpha_0| R + (1 - \rho)|\alpha_0| R + |\alpha_2 R - \alpha_1| R^2 + |\alpha_3 R^2 - \alpha_2 R^2| + |\alpha_4 R^3 - \alpha_3 R^3| + \ldots \]

\[ + |\alpha_{k-1} R - \alpha_{k-2} R^{k-1} + |\alpha_{k+1} R^k - \alpha_k R^k + |\alpha_{k+2} R^{k+1} - \alpha_{k+1} R^{k+1}| + \ldots \]

\[ + |a_{n-1} R^{n-1} + a_n R^n - (R - \mu) a_n R^n + |\alpha_n R^n + |a_n R^{n+1}| \]

\[ \leq 2|\alpha_0| R + \rho(\alpha_0) R + 2\epsilon_1 R^{k+1} + (|\alpha_n| - \alpha_n) R^{n+1} + \mu(\alpha_n + |\alpha_n|) R^n \]

\[ + 2 \sum_{j=0}^{n} \beta_j |R|^{j+1} = M. \]

Since \( F(z) \) is also analytic for \( |z| \leq R \), \( F(0) \neq 0 \), it follows by the lemma that the number of zeros of \( F(z) \) in \( |z| \leq \delta R, 0 < \delta < 1 \) is less than or equal to

\[ \frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|F(0)|} \leq \frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0| R}. \]

Since the zeros of \( P(z) \) are also the zeros of \( F(z) \), it follows that the number of zeros of \( P(z) \) in \( |z| \leq \delta R, 0 < \delta < 1 \) is less than or equal to

\[ \frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|F(0)|} \leq \frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0| R}. \]

That proves theorem 2.
Proof of Theorem 3: Consider the polynomial
\[ F(z) = (R - z)(P(z)) \]
\[ = (R - z)(a_0 + a_1 z + a_2 z^2 + \ldots + a_k z^{k-1} + a_k z^k + a_{k+1} z^{k+1} + \ldots + a_{n-1} z^{n-1} + a_n z^n) \]
\[ = a_0 R + (a_1 R - a_0) z + (a_2 R - a_1) z^2 + \ldots + (a_{k-1} R - a_k) z^{k-1} + (a_k R - a_{k-1}) z^k + (a_{k+1} R - a_k) z^{k+1} + \ldots + (a_{n-1} R - a_{n-2}) z^{n-1} + (a_n R - a_{n-1}) z^n \]
\[ = a_0 R + (\alpha_1 R - \alpha_0) z + (\alpha_2 R - \alpha_1) z^2 + \ldots + (\alpha_{k-1} R - \alpha_k) z^{k-1} + (\alpha_k R - \alpha_{k-1}) z^k + (\alpha_{k+1} R - \alpha_k) z^{k+1} + \ldots + (\alpha_{n-1} R - \alpha_{n-2}) z^{n-1} + (\alpha_n R - \alpha_{n-1}) z^n \]
\[ - \beta_n z^{n+1} + \sum_{i=0}^{n} \beta_i z^i + (\beta_{k+1} - \beta_k) z^k + \ldots + (\beta_{n-1} - \beta_{n-2}) z^{n-1} + (\beta_n - \beta_{n-1}) z^n \]
For \( |z| \leq R \), we have, by using the hypothesis
\[ |F(z)| \leq |a_0| R + |\alpha_1 R - \rho_1 \alpha_0 + \rho_1 \alpha_0 - \alpha_0| R + |\alpha_2 R - \alpha_1| R^2 + \ldots + |\alpha_{k-1} R - \alpha_k| R^{k-1} + |\alpha_k R - \alpha_{k-1}| R^k \]
\[ + |\beta_n R - \lambda \beta_n - \beta_{n-1}| R^k + |\beta_{n-1} - \beta_{n-2}| R^{n-1} + \ldots + |\beta_0 | R \]
\[ \leq |\alpha_0| R + (\alpha_1 R - \rho_1 \alpha_0) R + (1 - \rho_1) |\alpha_0| R + (\alpha_2 R - \alpha_1) R^2 + \ldots + (\alpha_{k-1} R - \alpha_k) R^{k-1} + (\alpha_k R - \alpha_{k-1}) R^k \]
\[ + (\alpha_{k+1} R - \alpha_k) R^{k+1} + \ldots + (\alpha_{n-2} R - \alpha_{n-1}) R^{n-1} \]
\[ + [(\alpha_{n-1} - (R - \mu) \alpha_n)] R^n + \mu |\alpha_n| R^n + \ldots + \mu |\alpha_n| R^{n-1} \]
\[ + (1 - \rho_1) |\beta_0| R + \ldots + (\beta_{n-1} R - \beta_{n-2}) R^{n-1} + \ldots + (\beta_{n-1} R - \beta_{n-2}) R^{n-1} \]
\[ \leq 2 |\alpha_0| R - \rho_1 R |\alpha_0| + \alpha_0 + 2 \alpha_2 R^{k+1} + \ldots + \mu |\alpha_n| R^n \]
\[ + \lambda |\alpha_n| R^n + \lambda |\beta_n| R^n \]
\[ = M \]
Since \( F(z) \) is also analytic for \( |z| \leq R \), \( F(0) \neq 0 \), it follows by Lemma 2 that the number of zeros of \( F(z) \) in \( |z| \leq \delta R, 0 < \delta < 1 \) is less than or equal to
\[ \frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0| R} = \frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|\alpha_0| R}. \]
Since the zeros of \( P(z) \) are also the zeros of \( F(z) \), it follows that the number of zeros of \( P(z) \) in \( |z| \leq \delta R, 0 < \delta < 1 \) is less than or equal to
$\frac{1}{\log \frac{1}{\delta}} \log M = \frac{1}{\log \frac{1}{\delta}} \log |a_0|R$.

That proves theorem 3.

V. REFERENCES


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