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SOME NEW RESULTS RELATED TO THE GENERALIZED SPECIAL FUNCTION OF FRACTIONAL CALCULUS

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ABSTRACT

In the present paper, we introduce two functions namely Ω(𝑐, 𝜖, 𝑝, 𝑞, 𝑧) and Ω(𝑐, 𝑣, 𝑝, 𝑞, 𝑧) in terms of Advanced M-Series [9] introduced recently by Sharma and show their properties by using fractional integrals and derivatives. Results derived in this paper are the extensions of the results derived earlier by Sharma and Dhakad [7].

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INTRODUCTION

The Advanced M-series [9] with 𝑝 + 2 upper parameters 𝑎₁, 𝑎₂, ..., 𝑎_𝑝, 𝑌, 𝜇 and 𝑞 + 1 lower parameters 𝑏₁, 𝑏₂, ..., 𝑏_𝑞, 𝛿 is

\[ \frac{\alpha}{\beta} \quad _p \quad M_q \left( a_1 \ldots a_p, \gamma, \mu ; b_1 \ldots b_q ; z \right) = \frac{\alpha}{\beta} \quad _p \quad M_q \left( x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k (\gamma)_k (\mu)_k \ldots (\delta)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k! \Gamma(\alpha k + \beta)} \]  

(1.1)

Here, 𝛼, 𝛽 ∈ ℂ, 𝑅(𝛼) > 0, 𝑚 > 0 and \((a_j)_k, (b_j)_k, (\gamma)_k, (\mu)_k,(\delta)_k \) are pochammer symbols. \( (n_k) > 0 \)
The series (1.1) is defined when none of the denominator parameters \( b_j, s, j = 1, 2, ... q \) is a negative integer or zero. If any parameter \( a_j \) is negative then the series (1.1) terminates into a polynomial in \( z \). By using ratio test, it is evident that the series (1.1) is convergent for all \( z \), when \( q ≥ p \), it is convergent for \( |z| < 1 \) when \( p = q + 1 \), divergent when \( p > q + 1 \). In some cases the series is convergent for \( z = 1, z = -1 \). Let us consider take,

\[ \beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j \]

when \( p = q + 1 \), the series is absolutely convergent for \( |z| = 1 \) if \( R(\beta) < 0 \), convergent for \( z = 1, \) if \( 0 \leq R(\beta) < 1 \) and divergent for \( |z| = 1, \) if \( 1 \leq R(\beta) \).

Some Special Cases

A) If we put \((\delta)_k = (\mu)_k, n_k = 1 \) in equation (1.2) it convertes in k

\[ _p \quad k \quad a, \beta, \gamma \left( a_1 \ldots a_p ; b_1 \ldots b_q ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k (\gamma)_k z^k}{(b_1)_k \ldots (b_q)_k} \frac{(\delta)_k}{k! \Gamma(\alpha k + \beta)} \]  

(1.2)


[3783]
B) If we put $(\delta)_k = (\mu)_k, n_k = 1, \gamma = 1$ in equation (1.2) it convertes in, Generalized M-Series [10]

$$\alpha, \beta \quad pM_q(z) = \sum_{k=0}^{\infty} \left( \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \right) \frac{z^k}{\Gamma(\alpha k + \beta)}$$

(1.3)

C) If we put $(\delta)_k = (\mu)_k, n_k = 1, \gamma = 1, \beta = 1$ in equation (1.2) it convertes in, M-Series [8]

$$\alpha \quad pM_q(z) = \sum_{k=0}^{\infty} \left( \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \right) \frac{z^k}{\Gamma(\alpha k + 1)}$$

(1.4)

D) $\quad pM_0$ i.e. no p upper or q lower parameters and $(\delta)_k = (\mu)_k, n_k = 1$

$$\alpha, \beta \quad 0M_0(z) = \sum_{k=0}^{\infty} \left( \frac{(\gamma)_k(z)^k}{\Gamma(\alpha k + \beta)(k)!} \right)$$

(1.5)

Thus the series reduced to the Mittag-Leffler function as in [6].

**MATHEMATICAL PREREQSITIES**

The Riemann-Liouville fractional integral of order $\nu \in \mathbb{C}$ is defined by Miller and Ross[3] (1993, p.45)

$$\quad 0D^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u)du,$$

(2.1)

where $\text{Re}(\nu)>0$. Following Samko et al. [6](1993, p. 37) we define the fractional derivative for $\alpha > 0$ in the form

$$\quad 0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(u)du}{(t-u)^{\alpha-n+1}}, \quad (n = \lfloor\text{Re}(\alpha)\rfloor + 1),$$

(2.2)

Where $\lfloor\text{Re}(\alpha)\rfloor$ means the integral part of $\text{Re}(\alpha)$.

Fractional Calculus Operators and **Advanced M-series**:

Let $f(t) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{(\gamma)_k(z)^k}{\Gamma(\beta)(k)!}$

**Fractional Integral and Fractional Derivative of the Sharma’s Advanced M-series [9]:**

Let us consider the fractional Riemann-Liouville (R-L) integral operator (for lower limit $a = 0$ with respect to variable $z$) of the **Advanced M-Series** (1.1).

$$I^\alpha_t f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (z-t)^{\nu-1} f(t) dt$$

$$= \frac{1}{\Gamma(v)} \int_{0}^{z} (z - t)^{v - 1} \sum_{k=0}^{\infty} \frac{(a_{1})_{k} \ldots (a_{p})_{k} \gamma_{k}(\mu)_{k}}{(b_{1})_{k} \ldots (b_{q})_{k} (\delta)_{k} (n_{k})!} (k)! (ct)^{k} dt$$

$$= \frac{1}{\Gamma(v)} \sum_{k=0}^{\infty} \frac{(a_{1})_{k} \ldots (a_{p})_{k} \gamma_{k}(\mu)_{k}(c)^{k}}{(b_{1})_{k} \ldots (b_{q})_{k} (\delta)_{k} (n_{k})!} (k)! (z - t)^{v - 1} t^{k} dt$$

$$= \frac{1}{\Gamma(v)} \sum_{k=0}^{\infty} \frac{(a_{1})_{k} \ldots (a_{p})_{k} \gamma_{k}(\mu)_{k}(c)^{k}}{(b_{1})_{k} \ldots (b_{q})_{k} (\delta)_{k} (n_{k})!} (k)! z^{k + 1 + v - 1} B(k + 1, v)$$

$$= \frac{1}{\Gamma(v)} \sum_{k=0}^{\infty} \frac{(a_{1})_{k} \ldots (a_{p})_{k} \gamma_{k}(\mu)_{k}(c)^{k}}{(b_{1})_{k} \ldots (b_{q})_{k} (\delta)_{k} (n_{k})!} (k)! z^{k + v} \frac{\Gamma(k + 1)\Gamma(v)}{\Gamma(k + 1 + v)}$$

$$= z^{v} \sum_{k=0}^{\infty} \frac{(a_{1})_{k} \ldots (a_{p})_{k} \gamma_{k}(\mu)_{k}(c)^{k}}{(b_{1})_{k} \ldots (b_{q})_{k} (\delta)_{k} (n_{k})!} (k)! \Gamma(k + 1 + v) (k)!$$

$$= z^{v} \sum_{k=0}^{\infty} \frac{(a_{1})_{k} \ldots (a_{p})_{k} \gamma_{k}(\mu)_{k}(c)^{k}}{(b_{1})_{k} \ldots (b_{q})_{k} (\delta)_{k} (n_{k})!} (k)! \Gamma(k + 1 + v) (k)!$$

We define \( \Omega(c, v, p, q, z) = z^{v} M_{q} (c z) \)

Analogously, \( R - L \) fractional derivative operator of the Advanced M-series \[9\] with respect to \( z \).

$$D_{z}^{\nu} f(z) = \frac{1}{\Gamma(n - \nu)} \left(\frac{d}{dz}\right)^{n} \int_{0}^{z} (z - t)^{n - \nu - 1} f(t) dt$$

$$= \frac{1}{\Gamma(n - \nu)} \left(\frac{d}{dz}\right)^{n} \int_{0}^{z} (z - t)^{n - \nu - 1} \sum_{k=0}^{\infty} \frac{(a_{1})_{k} \ldots (a_{p})_{k} \gamma_{k}(\mu)_{k}}{(b_{1})_{k} \ldots (b_{q})_{k} (\delta)_{k} (n_{k})!} (k)! (ct)^{k} dt$$

$$= \frac{1}{\Gamma(n - \nu)} \sum_{k=0}^{\infty} \frac{(a_{1})_{k} \ldots (a_{p})_{k} \gamma_{k}(\mu)_{k}(c)^{k}}{(b_{1})_{k} \ldots (b_{q})_{k} (\delta)_{k} (n_{k})!} (k)! (c)^{k} \left(\frac{d}{dz}\right)^{n} \int_{0}^{z} (z - t)^{n - \nu - 1} t^{k} dt$$

$$= \frac{1}{\Gamma(n - \nu)} \sum_{k=0}^{\infty} \frac{(a_{1})_{k} \ldots (a_{p})_{k} \gamma_{k}(\mu)_{k}(c)^{k}}{(b_{1})_{k} \ldots (b_{q})_{k} (\delta)_{k} (n_{k})!} (k)! (k)! \left(\frac{d}{dz}\right)^{n} z^{k + n - \nu} B(k + 1, n - \nu)$$

We use the modified Beta-function in above equation, which is defined as:

$$\int_{a}^{b} (b - t)^{\beta - 1} (t - a)^{\alpha - 1} dt = (b - a)^{\alpha + \beta - 1} B(\alpha, \beta),$$

Again, 

\[ D_2^v f(x) = \frac{1}{\Gamma(n-v)} \sum_{k=0}^{\infty} (a_1)_k \ldots (a_p)_k (y)_k (\mu)_k (c)_k^k \left( \frac{d}{dz} \right)^n z^{k+n-v} \frac{\Gamma(k+1)\Gamma(n-v)}{\Gamma(k+1+n-v)} \]  

(3.3)

Where \( k + 1 > 0, n - v > 0 \)

Differentiation \( n \) times the term \( z^{k+n-v} \) and using again \( \Gamma(a+k) = (a)_k \Gamma(a) \), representation (3.3) reduces to

\[
\begin{align*}
&= \sum_{k=0}^{\infty} (a_1)_k \ldots (a_p)_k (y)_k (\mu)_k \frac{\Gamma(k+n-v+1)(c)_k^k}{\Gamma(ak+\beta)(k+\beta)(k)\Gamma(k)} z^{k-v} \\
&= z^{-v} \sum_{k=0}^{\infty} (a_1)_k \ldots (a_p)_k (y)_k (\mu)_k (cz)_k^k \\
&= z^{-v} \sum_{k=0}^{\infty} (a_1)_k \ldots (a_p)_k (y)_k (\mu)_k (cz)_k^k \\
&= \alpha, \beta \\
D_2^v M_q (z) = z^{-v} M_q (cz) \\ 
\end{align*}
\]  

(3.4)

We define \( \Omega(c, -v, p, q, z) = z^{-v} M_q (cz) \)  

(3.5)

**PROPERTIES OF THE FUNCTIONS** \( \Omega(C, v, p, q, z) \) AND \( \Omega(C, -v, p, q, z) \)

Theorem 4.1 If \( c \) is an arbitrary constant then

\( I_2^g \Omega(c, v, p, q, z) = \Omega(c, \sigma + v, p, q, z) \)

Proof:

By the definition of the fractional integral (2.1) we have

\[
\begin{align*}
I_2^g \Omega(c, v, p, q, z) &= \frac{1}{\Gamma(\sigma)} \int_0^z (z-t)^{\sigma-1} \Omega(c, v, p, q, t) \ dt \\
&= \frac{1}{\Gamma(\sigma)} \int_0^z (z-t)^{\sigma-1} t^v \sum_{k=0}^{\infty} (a_1)_k \ldots (a_p)_k (y)_k (\mu)_k (ct)_k^k \Gamma(k+1+v)(k) \ dt \\
&= \frac{1}{\Gamma(\sigma)} \sum_{k=0}^{\infty} (a_1)_k \ldots (a_p)_k (y)_k (\mu)_k (cz)_k^k \int_0^z (z-t)^{\sigma-1} t^v (ct)_k^k \ dt \\
\end{align*}
\]

On Simplification and using Beta-function in above equation, we get the desired result

\[ I_z^\sigma \Omega( c, v, p, q, z) = \Omega( c, \sigma + v, p, q, z) \]

Theorem 4.2 If \( c \) is an arbitrary constant then

\[ D_z^\sigma \Omega( c, v, p, q, z) = \Omega( c, v - \sigma, p, q, z) \]

Proof: By the definition of the fractional derivative (2.2), we get

\[
D_z^\sigma \Omega( c, v, p, q, z) = \frac{1}{\Gamma(n-\sigma)} \left( \frac{d}{dz} \right)^n \int_0^\infty \frac{(a_1)_k \cdots (a_p)_k (y)_k (\mu)_k}{(b_1)_k \cdots (b_q)_k (\delta)_k (\nu)_k} \frac{(c\tau)_k}{\Gamma(\kappa+1)} d\tau
\]

\[ \Omega( c, v, p, q, z) = \frac{1}{\Gamma(n-\sigma)} \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k (y)_k (\mu)_k}{(b_1)_k \cdots (b_q)_k (\delta)_k (\nu)_k} \frac{(c\tau)_k}{\Gamma(\kappa+1)} \int_0^\infty \frac{(z - t)^{n-\sigma-1} t^{-v}}{\Gamma(n)} (\tau) \tau^k d\tau \]

Simplification and using Beta-function in above equation, we get the desired result

\[ D_z^\sigma \Omega( c, v, p, q, z) = \Omega( c, v - \sigma, p, q, z) \]

This completes the analysis.

REFERENCES