Abstract

Atanassov introduced the notion of intuitionistic fuzzy sets as a generalization of the notion of fuzzy sets. In this paper we introduce the concept of an intuitionistic \((\alpha, \beta)\)-fuzzy \(H_\alpha\)-subgroups of an \(H_\alpha\)-groups by using the notion of “belongingness \((\in)\)” and “quasi-coincidence \((q)\)” of fuzzy points with fuzzy sets, where \(\alpha \in \{\in, q\}\), \(\beta \in \{\in, q, \in \lor q, \in \land q\}\) and, then we investigate the basic properties of these notions.

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Introduction

The concept of hyperstructure was introduced in 1934 by Marty [1]. Hyperstructures have many applications to several branches of pure and applied sciences. Vougiouklis [2] introduced the notion of \(H_\alpha\)-structures, and Davvaz [3] surveyed the theory of \(H_\alpha\)-structures. After the introduction of fuzzy sets by Zadeh [4], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [5] is one among them. For more details on intuitionistic fuzzy sets, we refer the reader to [6, 7].

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [14], played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [8, 9] gave the concepts of \((\alpha, \beta)\)-fuzzy subgroups by using the notion of “belongingness \((\in)\)” and “quasi-coincidence \((q)\)” between a fuzzy point and a fuzzy subgroup, where \(\alpha, \beta\) are any two of \(\{\in, q, \in \lor q, \in \land q\}\) with \(\alpha \neq \in \land q\), and introduced the concept of an \((\in, \in \lor q)\)-fuzzy subgroup. In [10] Yuan, Li et al. redefined \((\alpha, \beta)\)-intuitionistic fuzzy subgroups. M. Asghari-Larimi [15] gave intuitionistic \((\alpha, \beta)\)-fuzzy \(H_\alpha\)-submodules. This paper continues this line of research for fuzzy \(H_\alpha\)-subgroups of \(H_\alpha\)-groups.

The paper is organized as follows: in section 2 some fundamental definitions on \(H_\alpha\)-structures and fuzzy sets are explored, in section 3 we define intuitionistic \((\alpha, \beta)\)-fuzzy \(H_\alpha\)-subgroups and establish some useful theorems.

Basic Definitions

We first give some basic definitions for proving the further results.

Definition 2.1 [11] Let \(X\) be a non-empty set. A mapping \(\mu : X \rightarrow [0, 1]\) is called a fuzzy set in \(X\).

Definition 2.2 [11] An intuitionistic fuzzy set \(A\) in a non-empty set \(X\) is an object having the form \(A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}\), where the functions \(\mu_A : X \rightarrow [0, 1]\) and \(\lambda_A : X \rightarrow [0, 1]\) denote the degree of membership and degree of non membership of each element \(x \in X\) to the set \(A\) respectively.

and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$. We shall use the symbol $A = \{\mu_A, \lambda_A\}$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$.

**Definition 2.3** [12] Let $H$ be a non-empty set and $*: H \times H \to \mathcal{P}^*(H)$ be a hyperoperation, where $\mathcal{P}^*(H)$ is the set of all the non-empty subsets of $H$. Where $A*B = \bigcup_{a \in A, b \in B} a*b$, $\forall A, B \subseteq H$.

The $*$ is called weak commutative if $x^*y \cap y^*x \neq \phi$, $\forall x, y \in H$.

The $*$ is called weak associative if $(x^*y)^*z \cap x^*(y^*z) \neq \phi$, $\forall x, y, z \in H$.

$(H, *)$ is called an $H_v$-group if

(i) $*$ is weak associative.

(ii) $a^*H = H^*a = H$, $\forall a \in H$ (Reproduction axiom).

**Definition 2.4** [13] Let $H$ be a hypergroup (or $H_v$-group) and let $\mu$ be a fuzzy subset of $H$. Then $\mu$ is said to be a fuzzy subhypergroup (or fuzzy $H_v$-subgroup) of $H$ if the following axioms hold:

(i) $\min\{\mu(x), \mu(y)\} \leq \inf_{x,y} \{\mu(x)\}$, $\forall x, y \in H$.

(ii) For all $x, a \in H$ there exists $\mu \in H$ such that $x \in a^*y$ and $\min\{\mu(a), \mu(x)\} \leq \mu(y)$.

**Definition 2.5** [5] Let $A = \{\mu_A, \lambda_A\}$ and $B = \{\mu_B, \lambda_B\}$ be intuitionistic fuzzy sets in $X$. Then $\mu$ is a fuzzy point if $\forall x, y \in X$ such that $x^*y \neq \phi$.

(iii) $\min\{\mu(x), \mu(y)\} \leq \inf_{x,y} \{\mu(x)\}$, $\forall x, y \in H$.

**Definition 2.6** [8] Let $\mu$ be a fuzzy subset of $R$. If there exist a $t \in (0, 1]$ and an $x \in R$ such that $\mu(x) = t$.

Then $\mu$ is called a fuzzy point with support $x$ and value $t$ and is denoted by $x_t$.

**Definition 2.7** [8] Let $\mu$ be a fuzzy subset of $R$ and $x_t$ be a fuzzy point.

(1) If $\mu(x) \geq t$, then we say $x_t$ belongs to $\mu$, and write $x_t \in \mu$.

(2) If $\mu(x) + t > 1$, then we say $x_t$ is quasi-coincident with $\mu$, and write $x_t \subseteq q^\mu$.

(3) $x_t \subseteq q^\mu \iff x_t \subseteq \mu$ or $x_t \subseteq q^\mu$.

(4) $x_t \subseteq \land q^\mu \iff x_t \subseteq \mu$ and $x_t \subseteq q^\mu$.

In what follows, unless otherwise specified, $\alpha$ and $\beta$ will denote any one of $\in, q, \in \lor q$ or $\in \land q$ with $\alpha \neq \in \land q$, which was introduced by Bhakat and Das [9].
Intuitionistic \((\alpha, \beta)\)-fuzzy \(H_v\)-subgroups

In this section we give the definition of intuitionistic \((\alpha, \beta)\)-fuzzy \(H_v\)-subgroup and prove some related results.

**Definition 3.1** Let \(H\) be a hypergroup (or \(H_v\)-group). An intuitionistic fuzzy set \(A = \{\mu_A, \lambda_A\}\) of \(H\) is called intuitionistic fuzzy subhypergroup (or intuitionistic fuzzy \(H_v\)-subgroup) of \(H\) if the following axioms hold:

1. \(\min\{\mu(x), \mu(y)\} \leq \inf_{a \in x \cdot y} \{\mu(a)\}, \quad \forall x, y \in H.\)
2. For all \(x, a \in H\) there exists \(y \in H\) such that \(x \in a \cdot y\) and \(\min\{\mu(a), \mu(x)\} \leq \mu(y).\)
3. \(\sup\{\lambda_a(\alpha)\} \leq \max\{\lambda_a(x), \lambda_a(y)\}, \quad \forall x, y \in H.\)
4. For all \(x, a \in H\) there exists \(y \in H\) such that \(x \in a \cdot y\) and \(\{\lambda_a(y)\} \leq \max\{\lambda_a(a), \lambda_a(x)\}.\)

**Definition 3.2** An intuitionistic fuzzy set \(A = \{\mu_A, \lambda_A\}\) in \(G\) is called an intuitionistic \((\alpha, \beta)\)-fuzzy \(H_v\)-subgroup of \(G\) if for all \(t, r \in (0, 1]\),

1. \(\forall x, y \in G, \quad x \cdot y, \alpha \mu_A \Rightarrow z_{t,r} \beta \mu_A\) for all \(z \in x \cdot y,\)
2. \(\forall x, a \in G, \quad x \cdot a, \alpha \mu_A \Rightarrow y_{t,r} \beta \mu_A\) for some \(y \in G\) with \(x \in a \cdot y,\)
3. \(\forall x, a \in G, \quad x \cdot a, \alpha \lambda_A \Rightarrow z_{t,r} \beta \lambda_A\) for all \(z \in x \cdot y,\)
4. \(\forall x, a \in G, \quad x \cdot a, \alpha \lambda_A \Rightarrow y_{t,r} \beta \lambda_A\) for some \(y \in G\) with \(x \in a \cdot y,\)

**Lemma 3.3** Let \(A = \{\mu_A, \lambda_A\}\) be an intuitionistic fuzzy set in \(G\). Then for all \(x \in G\) and \(r \in (0, 1]\), we have

1. \(x, q \mu_A \Leftrightarrow x, \in \mu_A^c;\)
2. \(x, q \mu_A \Leftrightarrow x, \in \land q \mu_A^c.\)

**Proof.** (1) Let \(x \in G\) and \(r \in (0, 1]\). Then, we have

\[x, q \mu_A \Leftrightarrow \mu_A(x) + t > 1 \]
\[\Leftrightarrow 1 - \mu_A(x) < t \]
\[\Leftrightarrow \mu_A^c(x) < t \]
\[\Leftrightarrow x, \in \mu_A^c.\]

(2) Let \(x \in G\) and \(r \in (0, 1]\). Then, we have

\[x, \in q \mu_A \Leftrightarrow x, \in \mu_A \quad \text{or} \quad x, q \mu_A \Leftrightarrow \mu_A(x) \geq t \quad \text{or} \quad \mu_A(x) + t > 1 \]
\[\Leftrightarrow 1 - \mu_A^c(x) \geq t \quad \text{or} \quad 1 - \mu_A^c(x) + t > 1 \]
\[\Leftrightarrow x, \in q \mu_A^c \quad \text{or} \quad x, \in \mu_A^c.\]

**Theorem 3.4** If \(A = \{\mu_A, \lambda_A\}\) is an intuitionistic \((\epsilon, \epsilon)\)-fuzzy \(H_v\)-subgroup of \(G\), then \(A = \{\mu_A, \lambda_A\}\) is an intuitionistic fuzzy \(H_v\)-subgroup of \(G\).


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Proof (1) Let \( x, y \in G \) and \( \mu_A(x) \wedge \mu_A(y) = t \). Then \( x_i, y_i \in \mu_A \). By condition (1) of definition 3.2, we have 
\[
\forall z \in x \cdot y, \quad \mu_A(z) \geq t, \quad \forall z \in x \cdot y.
\]
Consequently \( \mu_A(x) \wedge \mu_A(y) = t \leq \bigwedge_{x \cdot y} \mu_A(z) \) for all \( x, y \in G \).

(2) Now let \( x, a \in G \) and \( \mu_A(x) \wedge \mu_A(a) = t \). Then \( x_i, a_i \in \mu_A \). It follows from condition (2) of definition 3.2 that \( y_i \in \mu_A \), for some \( y \in G \) with \( x \in a \cdot y \).

Thus \( y_i \in \mu_A \), for some \( y \in G \) with \( x \in a \cdot y \).

So, for all \( x, a \in G \), there exist \( y \in G \) such that \( x \in a \cdot y \) and \( \mu_A(x) \wedge \mu_A(a) = t \leq \mu_A(y) \).

(3) Let \( x, y \in G \) and \( \lambda_A(x) \lor \lambda_A(y) = s \). If \( s = 1 \), then \( \lambda_A(z) \leq 1 = s \) for all \( z \in x \cdot y \). It is easy to see that
\[
\forall z \in x \cdot y. \quad \lambda_A(z) \leq \lambda_A(x) \lor \lambda_A(y)
\]
for all \( x, y \in G \).

If \( s < 1 \) there exists \( t \in (0, 1) \) such that \( \lambda_A(x) \lor \lambda_A(y) = s < t \).

Then \( x_i, y_i \in \lambda_A \). By condition (3) of definition 3.2, we have \( z_i \in \lambda_A \), \( \forall z \in x \cdot y \) and so \( \lambda_A(z) < t \).

Consequently \( \lambda_A(z) \leq \lambda_A(x) \lor \lambda_A(y) \) for all \( x, y \in G \).

(4) Now let \( x, a \in G \) and \( \lambda_A(x) \lor \lambda_A(a) = s \). If \( s < 1 \), there exists \( t \in (0, 1) \) such that \( \lambda_A(x) \lor \lambda_A(a) = s < t \).

Then \( x_i, a_i \in \lambda_A \). By condition (4) of definition 3.2, we have \( y_i \in \lambda_A \) for some \( y \in G \) with \( x \in a \cdot y \).

Hence \( \lambda_A(y) < t \) and \( \lambda_A(z) < t \).

Thus \( \lambda_A(y) \lor \lambda_A(z) < t \). This implies that for all \( x, a \in G \), there exist \( y \in G \) such that \( x \in a \cdot y \) and \( \lambda_A(y) \leq \lambda_A(x) \lor \lambda_A(a) \). If \( s = 1 \) the proof is obvious.

Theorem 3.5 If \( A = \{ \mu_A, \lambda_A \} \) is an intuitionistic \((\varepsilon, \varepsilon \lor q)\)-fuzzy \( H_v \)-subgroup of \( G \), then \( A = \{ \mu_A, \lambda_A \} \) is an intuitionistic fuzzy \( H_v \)-subgroup of \( G \).

Proof The proof is similar to the proof of Theorem 3.4.

Theorem 3.6 If \( A = \{ \mu_A, \mu^c_A \} \) is an intuitionistic \((\alpha, \beta)\)-fuzzy \( H_v \)-subgroup of \( G \) if and only if \( A = \{ \mu_A, \mu^c_A \} \) is an intuitionistic \((\alpha', \beta')\)-fuzzy \( H_v \)-subgroup of \( G \), where \( \alpha, \beta, \alpha' \in \{ \varepsilon, q \} \) and \( \beta \in \{ \varepsilon, q \} \).

Proof We only prove the case of \((\alpha, \beta) = (\varepsilon, \varepsilon \lor q)\). The others are analogous. Let \( A = \{ \mu_A, \mu^c_A \} \) be an intuitionistic \((\varepsilon, \varepsilon \lor q)\)-fuzzy \( H_v \)-subgroup of \( G \).

Condition (1). Let \( x, y \in G \) and \( t, r \in (0, 1) \) be such that \( x_i, y_i \in \mu_A \). It follows from Lemma 3.3 that \( z_i, y_i \in \mu^c_A \). Since \( \mu^c_A \) is an anti \((\varepsilon, \varepsilon \lor q)\)-fuzzy \( H_v \)-subgroup of \( G \). Thus by condition (3) of definition 3.2, we have
\[
\exists z_i \in \bigvee q \mu^c_A \text{ for all } z_i \in x \cdot y.
\]

By Lemma 3.3, this is equivalence with \( \exists z \in \wedge q \mu_A \text{ for all } z \in x \cdot y. \)

Thus condition of (1) of definition 3.2 is valid.
Condition (2). Suppose that \( x, a \in G \) and \( t, r \in (0, 1] \) be such that \( x, a, q \mu_A \). By Lemma 3.3, we have \( x, a, q \mu_A \) iff \( x, a, \in \mu_A \). By hypotheses, \( \mu_A \) is an anti \((\in, \in \lor \lor)\)-fuzzy \( H_v \)-subgroup of \( G \). Thus by condition (4) of definition 3.2, we have \( y_{r, r} \in \lor \mu_A \) for some \( y \in G \) with \( x \in a \cdot y \).

It follows from Lemma 3.2 that \( y_{r, r} \in \land \mu_A \) for some \( y \in G \) with \( x \in a \cdot y \).

Thus condition of (2) of definition 3.2 is valid.

Condition (3). Let \( x, y \in G \) and \( t, r \in (0, 1] \) be such that \( x, y, \bar{q} \mu_A \). It follows from Lemma 3.3 that \( x, y, \bar{q} \mu_A \) iff \( x, y \in \mu_A \). Since \( \Delta A = \{ \mu_A, \mu_A \} \) is an intuitionistic \((\in, \in \lor \lor)\)-fuzzy \( H_v \)-subgroup of \( G \). Thus by condition (1) of definition 3.2, we have \( z_{r, r} \in \lor \mu_A \) for all \( z \in x \cdot y \).

By Lemma 3.2, this is equivalence with \( z_{r, r} \in \land \mu_A \) for all \( z \in x \cdot y \).

Thus condition of (3) of definition 3.2 is valid.

Condition (4). Suppose that \( x, a \in G \) and \( t, r \in (0, 1] \) be such that \( x, a, q \mu_A \). This is equivalence with \( x, a, q \mu_A \). By hypotheses, \( \mu_A \) is an \((\in, \in \lor \lor)\)-fuzzy \( H_v \)-subgroup of \( G \). Thus by condition (2) of definition 3.2, we have \( y_{r, r} \in \lor \mu_A \) for some \( y \in G \) with \( x \in a \cdot y \).

It follows from Lemma 3.3 that \( y_{r, r} \in \land \mu_A \) for some \( y \in G \) with \( x \in a \cdot y \).

Thus condition of (4) of definition 3.2 is valid.

Theorem 3.7 If \( \Delta A = \{ \lambda_A, \lambda_A \} \) is an intuitionistic \((\alpha, \beta)\)-fuzzy \( H_v \)-subgroup of \( G \) if and only if \( \Delta A = \{ \lambda_A, \lambda_A \} \) is an intuitionistic \((\alpha', \beta')\)-fuzzy \( H_v \)-subgroup of \( G \), where \( \alpha \in \{ \in, \lor \} \)

and \( \beta \in \{ \in, \lor, \land \lor \} \).

**Proof** The proof is similar to the proof of Theorem 3.6.

Theorem 3.8 If \( A = \{ \mu_A, \lambda_A \} \) is an intuitionistic \((\alpha, \beta)\)-fuzzy \( H_v \)-subgroup of \( G \) if and only if \( \lambda_A \) is an \((\alpha, \beta)\)-fuzzy \( H_v \)-subgroup of \( G \) and \( \lambda_A \) is an \((\alpha', \beta')\)-fuzzy \( H_v \)-subgroup of \( G \), where \( \alpha \in \{ \in, \lor \} \)

and \( \beta \in \{ \in, \lor, \land \lor \} \).

**Proof** We only prove the case of \((\alpha, \beta) = (\in, \in \lor)\). The others are analogous. It is sufficient to show that, \( \lambda_A \) is an \((\in, \lor)\)-fuzzy \( H_v \)-subgroup of \( G \) if and only if \( \lambda_A \) is an anti \((\in, \in \lor)\)-fuzzy \( H_v \)-subgroup of \( G \). This is true, because \( x, q \lambda_A \iff x, \in \lambda_A \) and \( x, \in \land \lambda_A \iff x, \in \lor \lambda_A \), \( \forall x \in G \) and \( t \in (0, 1] \).

**References**


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