Zeros of the Sum of Two Polynomials

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Abstract
In this paper we find bounds for the zeros of the sum of two polynomials whose coefficients are restricted to certain conditions in the framework of Enestrom-Kakeya theorem.

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Introduction
A famous result giving a bound for all the zeros of a polynomial with real positive monotonically decreasing coefficients is the following result known as Enestrom-Kakeya theorem [2]:

Theorem A: Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that
\[
a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0.
\]
Then all the zeros of \( P(z) \) lie in the closed disk \( |z| \leq 1 \).

If the coefficients are monotonic but not positive, Joyal, Labelle and Rahman [1] gave the following generalization of Theorem A:

Theorem B: Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that
\[
a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0.
\]
Then all the zeros of \( P(z) \) lie in the closed disk
\[
|z| \leq \frac{a_n - a_0}{a_n}.
\]

If \( P(z) \) and \( Q(z) \) are two polynomials whose coefficients satisfy relations of the type (1), then the question arises whether Theorem A holds good for the sum \( P(z)+Q(z) \) of the two polynomials. It is easy to see that if the two polynomials have the same degree, then the conclusion of Theorem A holds good. But if \( P(z) \) and \( Q(z) \) are not of the same degree, then the result is not true. For example consider the polynomials \( P(z) = z^2 + z + 1 \) and \( Q(z) = 3z + 2 \). Both satisfy the conditions of Theorem A and both of them have their zeros in \( |z| \leq 1 \). But \( P(z)+Q(z) = z^2 + 4z + 3 \), whose zeros are -1, -3 with moduli 1 and 3 so that one zero does not lie in \( |z| \leq 1 \) and the theorem fails.

The same is true of the polynomials \( P(z) = 5z^2 + 4z + 3 \) and \( Q(z) = 8z + 7 \). The modulus of each zero of \( P(z)+Q(z) \) is \( 2 > 1 \). In such cases, however, we prove the following result:

Theorem 1: Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) and \( Q(z) = \sum_{j=0}^{m} b_j z^j \) be polynomials of degrees \( n \) and \( m \) \((n>m)\) respectively such that
\[
a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 \quad \text{and} \quad b_m \geq b_{m-1} \geq \ldots \geq b_1 \geq b_0.
\]
Then \( P(z)+Q(z) \) has all its zeros in the disk
\[
|z| \leq \frac{a_n + |b_m| + b_m + |a_0| + |b_0| - a_0 - b_0}{|a_n|}.
\]

The bound of Theorem 1 for the zeros of \( P(z)+Q(z) \) in the first example given above is 7 and \( |z| \leq 7 \) contains both -1 and -3. Similarly the bound for the second example is 4.2 and \( |z| \leq 4.2 \) contains both the zeros.

If we take \( Q(z)=0 \), Theorem 1 gives Theorem B.

Proof of Theorem

Proof of Theorem 1: Let \( n = m + p, \ p > 0 \) i.e. \( m = n - p \). Consider the polynomial

\[
F(z) = (1 - z)(P(z) + Q(z))
\]

\[
= (1 - z)[a_n z^n + a_{n-1} z^{n-1} + \ldots + a_{n-p+1} z^{n-p+1} + (a_{n-p} + b_{n-p}) z^{n-p} + \ldots
+ (a_{n-p-1} + b_{n-p-1}) z^{n-p-1} + \ldots + (a_1 + b_1) z + (a_0 + b_0)]
\]

\[
= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \ldots + (a_{n-p+2} - a_{n-p+1}) z^{n-p+2}
+ (a_{n-p+1} - a_{n-p} - b_{n-p}) z^{n-p+1} + (a_{n-p} + b_{n-p} - a_{n-p-1} - b_{n-p-1}) z^{n-p}
+ \ldots + (a_1 + b_1 - a_0 - b_0) z + (a_0 + b_0).
\]

For \( |z| > 1 \), we have, by using the hypothesis,

\[
|F(z)| \geq |a_n||z|^{n+1} - |(a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \ldots + (a_{n-p+2} - a_{n-p+1}) z^{n-p+2}
+ (a_{n-p+1} - a_{n-p} - b_{n-p}) z^{n-p+1} + (a_{n-p} + b_{n-p} - a_{n-p-1} - b_{n-p-1}) z^{n-p}
+ \ldots + (a_1 + b_1 - a_0 - b_0) z + (a_0 + b_0) | \\
\geq |a_n||z|^{n+1} - |(a_n - a_{n-1}) z^n| + |(a_{n-1} - a_{n-2}) z^{n-1}| + \ldots + |(a_{n-p+2} - a_{n-p+1}) z^{n-p+2}|
+ |(a_{n-p+1} - a_{n-p} - b_{n-p}) z^{n-p+1}| + |(a_{n-p} + b_{n-p} - a_{n-p-1} - b_{n-p-1}) z^{n-p}|
+ \ldots + |(a_1 + b_1 - a_0 - b_0) z| + |(a_0 + b_0) |
\]

\[
\geq |a_n| |z|^{n+1} - |a_n - a_{n-1}| |z|^n - |a_{n-1} - a_{n-2}| |z|^{n-1} - \ldots - |a_{n-p+2} - a_{n-p+1}| |z|^{n-p+2}
+ |a_{n-p+1} - a_{n-p} - b_{n-p}| |z|^{n-p+1} + |a_{n-p} + b_{n-p} - a_{n-p-1} - b_{n-p-1}| |z|^{n-p}
+ \ldots + |a_1 + b_1 - a_0 - b_0| |z| + |a_0 + b_0| \\
> |z|^n |a_n| |z| - (a_n - a_{n-1} - a_{n-2} - \ldots - a_{n-p+2} - a_{n-p+1} + a_{n-p} - a_{n-p-1} + \ldots + a_1 - a_0 - b_0)| \geq 0
\]

\[
|a_n| |z| - \{a_n + |b_{n-p}| + b_{n-p} + |a_0| + |b_0| - a_0 - b_0 \} > 0
\]

i.e.

\[
|z| > \frac{1}{|a_n|} (a_n + |b_{n-p}| + b_{n-p} + |a_0| + |b_0| - a_0 - b_0).
\]

This shows that those zeros of \( F(z) \) whose modulus is greater than 1 lie in

\[
|z| \leq \frac{1}{|a_n|} (a_n + |b_{n-p}| + b_{n-p} + |a_0| + |b_0| - a_0 - b_0).
\]

But the zeros of \( F(z) \) whose modulus is less than or equal to 1 already satisfy the above inequality. Therefore, it follows that all the zeros of \( F(z) \) lie in the disk

\[
|z| \leq \frac{1}{|a_n|} (a_n + |b_{n-p}| + b_{n-p} + |a_0| + |b_0| - a_0 - b_0).
\]

Since the zeros of \( P(z) \) are also the zeros of \( F(z) \), the result follows.
References
