Associators in the Nucleus of Antiflexible Rings

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Abstract

In this paper, first we prove that if R is a semi prime third power associative ring of char ≠ 2 then either N = C or R is associative. Using this result we prove that if R is a simple third power associative antiflexible ring of char ≠ 2,3 satisfying \((x, x, y) = k (y, x, x)\) for all \(x,y \in R\), \(k \neq 0\) and \(3k^2 + 2k + 1 \neq 0\) then either R is associative or nucleus equals center.

Keywords: Associator, commutator, nucleus, center, simple ring, prime ring.

Introduction

E. Kleinfeld and M. Kleinfeld [3] studied a class of Lie admissible rings. Also in [4] they have proved some results of a simple Lie admissible third power associative ring R satisfying an equation of the form \((x,y,x) = k(x,x,y)\) for all \(x,y \in R\), \(k \neq 0\) and \(k^2 + 2 \neq 0\). In this paper, we prove that if R is a simple third power associative antiflexible ring of char ≠ 2,3 satisfying \((x, x, y) = k (y, x, x)\) for all \(x,y \in R\), \(k \neq 0\) and \(3k^2 + 2k + 1 \neq 0\) then either R is associative or nucleus equals center.

Preliminaries

Let R be a non associative ring. We denote the commutator and the associator by \((x, y) = xy - yx\) and \((x,y,z) = (xy)z - x(yz)\) for all \(x,y,z \in R\) respectively. The nucleus \(N\) of a ring R is defined as \(N = \{ n \in R / (n,R) = (R,n,R) = (R,R,n) = 0 \}\). The center C of ring R is defined as \(C = \{ c \in N / (c,R) = 0 \}\). A ring R is called simple if \(R^2 \neq 0\) and the only non-zero ideal of R is itself. A ring R is called Prime if whenever A and B are ideals of R such that \(AB = 0\), then either \(A = 0\) (or) \(B = 0\).

Main Results

Let R be an antiflexible, then it satisfies the identity

\[ A(x, y, z) = (x, y, z) = (z, y, x) \]  \hspace{10pt} \text{---------(1)}

By the third power associativity, we have

\[ (x, x, x) = 0 \]  \hspace{10pt} \text{---------(2)}

Linearizing of (2) gives

\[ B(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0 \]  \hspace{10pt} \text{---------(3)}

We use the following two identities Teichmuller and semi Jacobi which holds in all rings

\[ C(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z = 0 \]  \hspace{10pt} \text{---------(4)}

And

\[ D(x, y, z) = (xy, z) - (x, yz) - (x, z)y - (x, y, z) - (z, x, y) + (x, z, y) = 0 \]  \hspace{10pt} \text{---------(5)}

We denote \( E(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) \)

\[ (x, y, z) + ((y, z), x) + ((z, x), y) = E(x, y, z) - E(x, z, y) \]  \hspace{10pt} \text{---------(6)}

Then from \(D(x, y, z) - D(y, x, z)\), we obtain

\[ ((x, y, z) + ((y, z), x) + ((z, x), y) = E(x, y, z) - E(x, z, y) \]  \hspace{10pt} \text{---------(7)}

As we observed by Maneri in [1], in any arbitrary ring with elements \(w, x, y, z\) we have

\[ 0 = C(w, x, y, z) - C(x, y, z, w) + C(y, z, w, x) - P(z, w, x, y) \]

\[ = E(wx,y,z) - E(xy z,w) + E(yz,w,x) - E(zw,x,y) - (w, x, yz) + (x, y, z) + (y, z, w) + (z, w, x) \]

We now assume that R satisfies identity (3), \(E(a, b, c) = 0\) for all \(a, b, c \in R\).

So the above equation imply

\[ (w, (x, y, z)) + (x, (y, z, w)) + (z, (w, x, y)) = 0 \]  \hspace{10pt} \text{---------(8)}

Let N be the nucleus of R and let \(n \in N\). By substituting \(n\) for \(w\) in (8), we get

\[ (n, (x, y, z)) = 0 \]  \hspace{10pt} \text{---------(9)}

i.e., \(n\) commutes with all associators.

The combination of (9) & (4) yields

\[(n, w(x, y, z)) = - (n, (w, x, y)z) \]  \(-----(10)\)

If \(u\) and \(v\) are two associators in \(R\), then substituting \(z=n, x=u, y=v\) in (5), we get

\[(uv, n) = 0 \]  \(-------(11)\)

If \(u = (a, b, c)\) then

\[(a(b, c, v), n) = 0 \]  \(------(12)\)

Now we prove the following theorem.

**Theorem:** If \(R\) is semiprime third power associative ring of char \(\neq 2\) which satisfies (3), then either \(N = C\) (or) \(R\) is associative.

**Proof:**

If \(N \neq C\), then there exist \(n \in N\) and \(a \in R\) such that \((a, n) \neq 0\).

Hence from (12), we have \((b, c, v) = 0\) for all associators \(v\) and \(b, c \in R\).

We can write this as \((R, R, (R, R, R))) = 0\).

By putting \(v = (q, r, s)\) in (11), we get \((u(q, r, s), n) = 0\)

Using (10) this leads to \(-((u, q, r)s, n) = 0\).

From this and (5), we obtain \((u, q, r) (s, n) = 0\).

Since \(N \neq C\), we have \((u, q, r) = 0\) for all associators \(u, q, r \in R\).

We can write this as \(((R, R, R), R, R) = 0\).

Using \((R, R, (R, R, R))) = 0 = ((R, R, R), R, R)\) the identity (3) gives

\[(R, (R, R, R), R) = 0 \]

Thus \((R, R, R) \subseteq N\). Since \(R\) is semiprime, we use the result in [1] to conclude that \(R\) must be associative.

This completes the proof of the theorem.

Henceforth we assume that \(R\) satisfies an equation of the form

\[(x, y, z) = k (y, x, x) \]  \(----- (13)\)

for all \(x, y \in R\), \(k \neq 0\) and using (3), identity (13) implies

\[(x, y, z) + (y, x, z) + (z, x, y) = 0 \]

Putting \(y = x, z = y \Rightarrow (x, x, y) + (x, y, x) + (y, x, x) = 0\)

\[k(y, x, x) + (x, y, x) + (y, y, x) = 0 \]

\[(k+1) (y, x, x) + (x, y, x) = 0 \]

\[(x, x, y) = -(k+1) (y, x, x) \]

\[= -(k+1) (x, x, y) \]

\[= -(\frac{k+1}{k}) (x, x, y) \]  (by (1) & (13)) \(-------(14)\)

**Lemma:** Let \(T = \{ t \in R / (t, N) = 0 = (tR, N) = (Rt, N) \}\). Then \(T\) is an ideal of \(R\).

**Proof:** Let \(t \in T, n \in N\) and \(x, y, z \in R\).

Then \((t, x, y, n) = (t, y, x, n) = 0\).

Using (9) and the definition of \(T\). Also (5) implies \((y, x, n) = (y, x, n)tx\).

But (5) also yields \((yt, n) = (y, n)t = 0 \) since \((yt, n) = 0\).

Now, \((y, n), t, x) = ((y, n), t)x - (y, n), tx\)

\[= 0 - (y, n), tx \]

\[= -(y, x, n) \]  \(----------(15)\)

or \((y, x, n) = -(y, x, n), t\)

\[= -(y, x, n), t \]  \(----------(16)\)

Using (14), \((x, x, y) = k (y, x, x)\)

While \(0 = C (x, x, y, n) = (x, y, x, y)n \) and \(0 = C (y, x, x, n) = (y, x, x, n) - (y, x, x)\)

Substitute this in (16) and using (14) & (9) gives

\[(x, y, x)n = -(y, x, x) - (y, x, x)n \]

\[= (x, y)n - n(y, x, x) \]  \(by (1)\)

\[= (x, y)n - n(x, x, y) \]  \(by (4)\)

\[= (x, x, y)n \]

\[= 0 \]

Linearizing the above identity, we get


[1007-1011]
(y, n, x, z) = -(y, n, z, x) \hspace{1cm} \text{(17)}

Again consider \((x, n), y, x\) = \((x, y, n) - (x, n, y)\) \hspace{1cm} \text{(18)}

From \(C(x, n, y, x) = 0\), it follows that \(x(n, y, x) = (x, n, y, x)\)

From (14), it follows that

\[(x, n, y, x) = -(k+1)(x, x, n) \hspace{1cm} \text{(by (14))}\]

\[= -(k+1)(ny, x, x) \hspace{1cm} \text{(by (1))}\]

And from \(C(n, y, x, x) = 0\) we have

\[(ny, x, x) = n(y, x, x) \hspace{1cm} \text{(9)}\]

Thus we have \(x(n, y, x) = -(k+1)(x, x, n) \hspace{1cm} \text{(19)}\)

From (9), it follows that \(x(n, y, x) = n(x, y, x)\)

From (14), we have \(n(x, y, x) = -(k+1)n(x, x, y)\)

Therefore \(x(n, y, x) = n(x, y, x) = -(k+1)n(x, x, y) \hspace{1cm} \text{(20)}\)

Substitute (19) and (20) in (18), we get

\[(x, n), y, x] = (xn, y, x) - nx, y, x] \hspace{1cm} \text{(21)}

Linearizing (21), we get

\[(x, n), y, x] = -(k+1)n(x, x, y) + (k+1)n(x, x, y) \hspace{1cm} \text{(22)}

Combining (17) and (22), we get

\[(\pi(x), n), \pi(y), \pi(z)) = \text{sgn}(\pi)((x, n), y, z) \hspace{1cm} \text{(23)}

for every permutation \(\pi\) on the set \{x, y, z\}.

Applying (23), we see that \( (y, n, t, x) = (y, n), x, t) \hspace{1cm} \text{(by (17))}\)

\[= -(t, n), x, y) \hspace{1cm} \text{(by (22))}\]

\[= (t, n), y, x) \hspace{1cm} \text{(by (17))}\]

\[= 0 \hspace{1cm} \text{(by defn. of T)}\]

Combined this with (15), we obtain \((y, x, n) = 0\)

So \(T\) is a right ideal of \(R\). By using the anti-isomorphic ring, we similarly prove that \(T\) is a left ideal of \(R\).

Therefore \(T\) is an ideal of \(R\).

**Theorem:** If \(R\) is a simple third power associative antiflexible ring with (13) of char \(\neq 2,3\) is either associative or satisfies nucleus equals center, \(N = C\).

**Proof:** Simplicity of \(R\) implies either that \(T = R\) or \(T = 0\).

If \(T = R\), then \(N = C\).

Hence assume that \(T = 0\).

Let \(u = (a, b, c)\) be an arbitrary associator with elements \(a, b, c \in R\).

We have already observed that for every associator \(v\), we have \((uv, n) = 0\).

Now using \((C(u, x, y, n) = 0)\) and (9) gives

\[(u, x, y), n) = -(u(x, x, y), n) = 0 \hspace{1cm} \text{(19)}\]

Using \((C(x, y, x, u), n) = 0\) gives \((y(x, x, u), n) = -(y(x, x, u)n, n) = 0 \hspace{1cm} \text{(14)}\)

Also \(y(x, x, u) = k(y(x, x, x) \hspace{1cm} \text{by (14)}\)

\[\Rightarrow \quad y(u, x, x) = \frac{1}{k}y(x, x, u) \hspace{1cm} \text{by (7)}\]

So \((y(u, x, x), n) = 0 \hspace{1cm} \text{Since (u, x, n), n) = 0 (by (7))}\)

We have \((u, x, x) \in T\).

Since we are assuming \(T = 0\), we have \((u, x, x) = 0\) for all \(x \in R\).

Using this in (14), we get

\[(x, u, x) = 0 \hspace{1cm} \text{and (x, x, u) = 0} \hspace{1cm} \text{(24)}\]

For \(a, b \in R\), we define \(a \equiv b\) if and only if \((a-b, n) = 0\) for all \(n \in N\).

Let \(a = x(y, x, z)\)

Because of (9), all associators are congruent to zero.

Thus \(C(x, y, x, z) = 0 \hspace{1cm} \text{Implies a = -(x, y, x)z}\).

Equ (14) implies \(a = -(x, y, x)z = (k+1)(y, x, x)z\)

By using \(C(w, x, y, z) = 0\) continuously and (14) yields

\[a = x(y, x, z) \hspace{1cm} \text{by (14)}\]

\[= -(x, y, x)z \hspace{1cm} \text{(by (14))}\]

\[\equiv (k+1)z \hspace{1cm} \text{(by (14))}\]

\[\equiv (k+1)z \hspace{1cm} \text{(by (14))}\]
\[ \equiv - \left( \frac{k+1}{k} \right) x(x, y, z) \]
\[ \equiv \left( \frac{k+1}{k} \right) (x, y, z) \]
\[ \equiv (k+1) (y, x, z) \]
\[ = \left( \frac{k+1}{k} \right) y(x, z, x) \]
\[ = k y(x, z, x) \]
\[ = k y(x, z, x) \]
\[ \equiv k (k+1) y(x, z, x) \]
\[ \equiv k (k+1) (y, z, x) \]  \[ \cdots (25) \]

Permuting \( y \) and \( z \) in (25), we get
\[ \beta = x(z, x, y) = - (x, y, z) \]
\[ = \left( \frac{1}{k+1} \right) (x, x, z) \]
\[ = - \left( \frac{k+1}{k} \right) x(x, z, x) \]
\[ = (k+1) (z, x, x) \]
\[ = - k (z, x, x) \]
\[ = k (z, x, x) \]
\[ = k (k+1) z(x, x, y) \]
\[ \beta = k (k+1) (z, x, x) \]
\[ \cdots (26) \]

From identity (3) we obtain
\[ x(x, y, z) + x(z, x, y) = - x(y, z, x) \]  \[ \cdots (27) \]

Using (25) and (26) in (27), we get
\[ \left( \alpha - \frac{k}{k+1} \right) + \beta = - x(y, z, x) \]  \[ \cdots (28) \]

However \( C(x, y, z, x) = 0 \) gives \(-x(y, z, x) \equiv (x, y, z)\)

Thus \( \left( \alpha - \frac{k}{k+1} \right) + \beta = (x, y, z) \)  \[ \cdots (29) \]

However using (1) and \( C(z, x, x, x) = 0 \), we have
\[ (x, x, z)x = (z, x, x)x \]  \[ \text{(by (1))} \]
\[ = - z(x, x, x) \]
\[ = 0 \]

Since \( (x, x, x) = 0 \), we have \( (x, x, z)x = 0 \)

Linearization of this gives
\[ (x, y, z)x + (y, x, z)x + (x, x, z)y \equiv 0 \]
\[ \text{or } (x, y, z)x \equiv -(y, x, z)x - (x, x, z)y \]  \[ \cdots (30) \]

Using (29), (25) and (26) in (30), we get
\[ \left( \frac{k}{k+1} \right) + \beta = \alpha + \beta - \frac{k}{k+1} \beta \]
\[ \frac{\alpha + \beta}{k+1} = \beta \]
\[ \left( \frac{2k^2 + k + 1}{k+1} \right) \alpha = \left( \frac{k^2 + k + 1}{k+1} \right) \beta \]

Using (25) and (26) to substitute for \( \frac{\alpha}{k+1} \) and \( \frac{\beta}{k+1} \) in the above equation gives
\[ \left( \frac{k^2 + k + 1}{k+1} \right) \left( \frac{1}{k} \right) x(x, y, z) \equiv (2k+1)(x, y, z) \]
\[ \Rightarrow (k^2 + k + 1) x(x, y, z) \equiv (2k^2 + k) x(x, y, z) \]  \[ \cdots (31) \]

Linearizing (31), we obtain
\[ (k^2 + k + 1) (w(x, y, z) + x(w, y, z)) \equiv (2k^2 + k)(w(x, y, z) + x(w, y, z)) \]  \[ \cdots (32) \]

By substituting \( w = u = (a, b, c) \) in (32) and using (11), we get
\[ (k^2 + k + 1) x(u, y, z) \equiv (2k^2 + k) x(u, y, z) \]  \[ \cdots (33) \]

Linearizing (24) we have \((u, z, y) = - (u, y, z)\)

Using this in (33), we obtain
\[ (k^2 + k + 1) x(u, y, z) \equiv - (2k^2 + k) x(u, y, z) \]
\((3k^2+2k+1)\ x(u, y, z) = 0\)

Thus if \((3k^2+2k+1) \neq 0\), we have \(x(u, y, z) \equiv 0\)

or \([x(u, y, z), n] = 0\) for all \(n \in \mathbb{N}\)

Thus \((u, y, z) \in T\). Since \(T = 0\), we have \((u, y, z) = 0\)

Similarly, \((\frac{k^2+k+1}{k(k+1)})\ \alpha = (\frac{2k+1}{k+1})\ \beta\) also yields

\((k^2+k+1)\ (y, z, x)x \equiv (2k^2 + k)\ (z, y, x)x\)

Linearizing the above equation, we get

\((k^2+k+1)(y, z, x)^w + (y, z, w)x) = (2k^2 + k)(z, y, x)^w + (z, y, w)x\)

Putting \(w = u = (a, b, c)\) in above and using (11), using \((z, y, x)u = 0\) and \((y, z, x)u = 0\), we get

\((k^2+k+1)(y, z, u)x \equiv (2k^2 + k)(z, y, u)x\)

Linearizing (24), we have \((z, y, u) = -(y, z, u)\)

using this in the previous equ, we obtain

\((3k^2+2k+1)(y, z, u)x \equiv -(2k^2 + k)(y, z, u)x\)

\(\Rightarrow\)

\((3k^2+2k+1)(y, z, u)x = 0\)

Thus if \((3k^2+2k+1) \neq 0\), we have \((y, z, u)x \equiv 0\)

or \([y, z, u], n] = 0\) for all \(n \in \mathbb{N}\)

Using \(C(x, y, z, u) = 0\) and \((x, y, z)u = 0\)

\(x(y, z, u) = 0\) or \((x(y, z, u), n] = 0\) for all \(n \in \mathbb{N}\).

Thus \((y, z, u) \in T\). Since \(T = 0\) we have \((y, z, u) = 0\)

Now we have both \((y, z, u) = 0\) and \((u, y, z) = 0\).

Using these two equations in (3) we get \((z, u, y) = 0\)

Now we are in the situation where all associators are in the nucleus.

i.e., \((R, R, R) \in \bigcap N\).

we use result in [2] to conclude that R must be associative.

**References**


