An Extension of Enestrom-Kakeya Theorem

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Abstract
In this paper we give an extension of the famous Enestrom-Kakeya Theorem, which generalizes many generalizations of the said theorem as well.

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Introduction
A famous result giving a bound for all the zeros of a polynomial with real positive monotonically decreasing coefficients is the following result known as Enestrom-Kakeya theorem [4]:

Theorem A: Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that
\[
a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0 .
\]
Then all the zeros of \( P(z) \) lie in the closed disk \( |z| \leq 1 \).

If the coefficients are monotonic but not positive, Joyal, Labelle and Rahman [3] gave the following generalization of Theorem A:

Theorem B: Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that
\[
a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 .
\]
Then all the zeros of \( P(z) \) lie in the closed disk \( |z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|} \).

Aziz and Zargar [1] generalized Theorem B by proving the following result:

Theorem C: Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that for some \( k \geq 1 \),
\[
k a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 .
\]
Then all the zeros of \( P(z) \) lie in the closed disk
\[
|z + k - 1| \leq \frac{k a_n - a_0 + |a_0|}{|a_n|} .
\]

Gulzar [2] generalized Theorem C to polynomials with complex coefficients and proved the following results:

Theorem D: Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with \( \text{Re}(a_j) = \alpha_j \),
\[
\text{Im}(a_j) = \beta_j, \quad j = 0,1,\ldots,n \quad \text{such that for some} \quad k \geq 1, 0 < \tau \leq ,
\]
\[k\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_1 \geq \tau \alpha_0.\]

Then all the zeros of \(P(z)\) lie in the closed disk
\[|z + (k - 1)\frac{\alpha_n}{a_n}| \leq \frac{k\alpha_n + 2|\alpha_0| - \tau (|\alpha_0| + |\alpha_0|) + 2 \sum_{j=0}^{n} |\beta_j|}{|a_n|}.\]

**Theorem E:** Let \(P(z) = \sum_{j=0}^{n} a_j z^j\) be a polynomial of degree \(n\) with \(\text{Re}(a_j) = \alpha_j\).

\[\text{Im}(a_j) = \beta_j, \quad j = 0, 1, \ldots, n\]

such that for some \(k \geq 1, 0 < \tau \leq 1\),
\[k\beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \tau \beta_0.\]

Then all the zeros of \(P(z)\) lie in the closed disk
\[|z + (k - 1)\frac{\beta_n}{a_n}| \leq \frac{k\beta_n + 2|\beta_0| - \tau (|\beta_0| + |\beta_0|) + 2 \sum_{j=0}^{n} |\alpha_j|}{|a_n|}.\]

Recently, Liman and Shah [5] proved the following generalization of Theorem C for polynomials having real coefficients:

**Theorem F:** Let \(P(z) = \sum_{j=0}^{n} a_j z^j\) be a polynomial of degree \(n\) with real coefficients such that for some \(t > 0\) and \(1 \leq \lambda \leq n\),
\[a_0 \leq a_1 \leq \ldots \leq a_{\lambda-1} \leq t a_{\lambda} \leq t^2 a_{\lambda+1} \leq \ldots \leq t^{n-\lambda} a_{n-1} \leq t^{n-k+1} a_n.\]

Then all the zeros of \(P(z)\) lie in
\[|z + t - 1| \leq \frac{a_n - a_0 + |a_0| + (t - 1)\sum_{j=\lambda}^{n} (a_j + |a_j|) - |a_n|}{|a_n|}.\]

The aim of this paper is to apply Theorem F to polynomials with complex coefficients and prove

**Theorem 1:** Let \(P(z) = \sum_{j=0}^{n} a_j z^j\) be a polynomial of degree \(n\) with \(\text{Re}(a_j) = \alpha_j\),
\[\text{Im}(a_j) = \beta_j, \quad j = 0, 1, \ldots, n\]

such that for some \(t > 0\), \(k \geq 1, 0 < \tau \leq 1\) and \(1 \leq \lambda \leq n\),
\[\tau \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_{\lambda-1} \leq t \alpha_{\lambda} \leq t^2 \alpha_{\lambda+1} \leq \ldots \leq t^{n-\lambda} \alpha_{n-1} \leq kt^{n-k+1} \alpha_n.\]

Then all the zeros of \(P(z)\) lie in
\[|z + (kt - 1)\frac{\alpha_n}{a_n}| \leq \frac{kt \alpha_n + 2|\alpha_0| - \tau (|\alpha_0| + |\alpha_0|) + (t - 1) \sum_{j=\lambda}^{n} (a_j + |a_j|) + 2 \sum_{j=0}^{n} |\beta_j|}{|a_n|}.\]

If \(a_j\) is real i.e. \(\beta_j = 0, \forall j = 0, 1, \ldots, n\), we immediately get the following result:

**Corollary 1:** Let \(P(z) = \sum_{j=0}^{n} a_j z^j\) be a polynomial of degree \(n\) such that for some \(t > 0\), \(k \geq 1, 0 < \tau \leq 1\) and \(1 \leq \lambda \leq n\),
\[\alpha_0 \leq a_1 \leq \ldots \leq a_{\lambda-1} \leq t a_{\lambda} \leq t^2 a_{\lambda+1} \leq \ldots \leq t^{n-\lambda} a_{n-1} \leq kt^{n-k+1} a_n.\]


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Then all the zeros of \( P(z) \) lie in

\[
|z + kt - 1| \leq \frac{kta_n + 2|a_0| - \tau (a_0 + |a_0|) + (t-1) \sum_{j=k}^{n-1} (a_j + |a_j|)}{|a_n|}.
\]

**Remark 1:** For \( k=1, \tau = 1 \), Cor. 1 reduces to Theorem F. For \( t=1 \), Theorem 1 reduces to Theorem D. Applying Theorem 1 to the polynomial \(-iP(z)\), we get the following result:

**Corollary 2:** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with \( \text{Re}(a_j) = \alpha_j \), \( \text{Im}(a_j) = \beta_j, j = 0, 1, \ldots, n \) such that for some \( t>0 \), \( k \geq 1, 0 < \tau \leq 1 \) and \( 1 \leq \lambda \leq n \),

\[
\tau \beta_0 \leq \beta_1 \leq \ldots \leq \beta_{\lambda-1} \leq t \beta_\lambda \leq t^2 \beta_{\lambda+1} \leq \ldots \leq t^{n-2} \beta_{n-1} \leq kt^{n-k+1} \beta_n.
\]

Then all the zeros of \( P(z) \) lie in

\[
|z + kt - 1| \leq \frac{kt \beta_n + 2|\beta_0| - \tau (\beta_0 + |\beta_0|) + (t-1) \sum_{j=\lambda}^{n-1} (\beta_j + |\beta_j|) + 2 \sum_{j=0}^{n} |\alpha_j|}{|a_n|}.
\]

**Remark 2:** For \( t=1 \), Theorem 1 reduces to Theorem E. Taking \( k=1 \), we get the following result from Theorem 1:

**Corollary 3:** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with \( \text{Re}(a_j) = \alpha_j \), \( \text{Im}(a_j) = \beta_j, j = 0, 1, \ldots, n \) such that for some \( t>0 \), \( 0 < \tau \leq 1 \) and \( 1 \leq \lambda \leq n \),

\[
\tau \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_{\lambda-1} \leq t \alpha_\lambda \leq t^2 \alpha_{\lambda+1} \leq \ldots \leq t^{n-2} \alpha_{n-1} \leq t^{n-k+1} \alpha_n.
\]

Then all the zeros of \( P(z) \) lie in

\[
|z + (t-1) \frac{\alpha_n}{a_n}| \leq \frac{t \alpha_n + 2|\alpha_0| - \tau (\alpha_0 + |\alpha_0|) + (t-1) \sum_{j=\lambda}^{n-1} (\alpha_j + |\alpha_j|) + 2 \sum_{j=0}^{n} |\beta_j|}{|a_n|}.
\]

For other different values of the parameters in the above results, we get many other interesting results.

**Proofs of Theorems**

**Proof of Theorem 1:** Consider the polynomial \( F(z) = (1-z)P(z) \)

\[
= (1-z)(a_n z^n + a_{n-1} z^{n-1} + \ldots + a_{\lambda+1} z^{\lambda+1} + a_{\lambda} z^{\lambda} + a_{\lambda-1} z^{\lambda-1} + \ldots + a_{n-1} z^{n-1} + a_n z^n)
\]

\[
= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \ldots + (a_{\lambda+1} - a_{\lambda}) z^{\lambda+1} + (a_{\lambda} - a_{\lambda-1}) z^{\lambda} + (a_{\lambda-1} - a_{\lambda-2}) z^{\lambda-1} + \ldots + (a_1 - a_0) z + a_0
\]

\[
= -a_n z^{n+1} + \{k(t \alpha_\lambda - \alpha_{n-1}) - (t \alpha_{n-1} - \alpha_n)\} z^n + \{t \alpha_{n-1} - \alpha_n\} z^{n-1} + \ldots + \{(t \alpha_{\lambda+1} - \alpha_{\lambda}) - (t \alpha_{\lambda} - \alpha_{\lambda+1})\} z^{\lambda+1} + \{t \alpha_{\lambda} - \alpha_{\lambda+1}\} z^{\lambda} + (\alpha_{\lambda-1} - \alpha_{\lambda-2}) z^{\lambda-1} + \ldots + \{(\alpha_1 - \alpha_0) + (t \alpha_0 - \alpha_0)\} z + a_0
\]

\[
+ i \sum_{j=1}^{n}(\beta_j - \beta_{j-1}) z^j + \beta_0 \}
\]


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For $|z| > 1$, we have by using the hypothesis
\[
|F(z)| \geq |a_n z^{n+1} + (kt-1)\alpha_n z^n - (kt\alpha_n - \alpha_{n-1})z^n + (t\alpha_{n-1} - \alpha_{n-2})z^{n-1} - (t-1)\alpha_{n-1}z^{n-1} \\
+ \ldots + (t\alpha_{j+1} - \alpha_j)z^{j+1} - (t-1)\alpha_jz^j + (t\alpha_j - \alpha_{j-1})z^j - (t-1)\alpha_jz^j \\
+ (\alpha_{j-1} - \alpha_{j-2})z^{j-1} + \ldots + \{(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)\}z + \alpha_0 \\
+ t\sum_{j=1}^n (\beta_j - \beta_{j-1})z^j + \beta_0 \} |
\]
\[
\geq |z|^n \left[ |a_n z + (kt-1)\alpha_n| - \left\{kt\alpha_n - \alpha_{n-1} + \frac{t\alpha_{n-1} - \alpha_{n-2}}{|z|} + \frac{t-1}{|z|^2} \alpha_{n-1} + \ldots \\
+ \frac{t\alpha_{j+1} - \alpha_j}{|z|^{n-j}} + \frac{t-1}{|z|^{n-j}} \alpha_j + \ldots + \frac{t\alpha_j - \alpha_{j-1}}{|z|^{n-j}} + \frac{t-1}{|z|^{n-j}} \alpha_{j-1} - \alpha_{j-2} + \ldots \\
+ \alpha_1 - \tau\alpha_0 + (1-\tau)\alpha_0 + \alpha_0 + \sum_{j=1}^n (|\beta_j| + |\beta_{j-1}|) + |\beta_0| \right\} \right]
\]
\[
\geq |z|^n \left[ |a_n z + (kt-1)\alpha_n| - \left\{kt\alpha_n + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) + (t-1)\sum_{j=1}^{n-1} (\alpha_j + |\alpha_j|) \\
+ 2\sum_{j=0}^n |\beta_j| \right\} \right]
\]

if
\[
|a_n z + (kt-1)\alpha_n| \geq \left\{kt\alpha_n + 2\alpha_0| - \tau(\alpha_0 + |\alpha_0|) + (t-1)\sum_{j=1}^{n-1} (\alpha_j + |\alpha_j|) \\
+ 2\sum_{j=0}^n |\beta_j| \right\}
\]

This shows that the zeros of $F(z)$ of modulus greater than 1 lie in
\[
\left| z + (kt-1)\frac{\alpha_n^j}{a_n} \right| \leq \frac{kt\alpha_n + 2\alpha_0 - \tau(\alpha_0 + |\alpha_0|) + (t-1)\sum_{j=1}^{n-1} (\alpha_j + |\alpha_j|) + 2\sum_{j=0}^n |\beta_j|}{|a_n|}.
\]

Since the zeros of $F(z)$ less than or equal to 1 also satisfy the above inequality, it follows that all the zeros of $F(z)$ lie in


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Since the zeros of $P(z)$ are also the zeros of $F(z)$, it follows that all the zeros of $P(z)$ lie in

$$
\left| z + (kt - 1) \frac{\alpha_n}{a_n} \right| \leq \frac{kt \alpha_0 + 2|\alpha_0| - \tau(|\alpha_0| + |\alpha_0|) + (t - 1) \sum_{j=\lambda}^{n-1} (\alpha_j + |\alpha_j|) + 2 \sum_{j=0}^{n} |\beta_j|}{|a_n|}.
$$

That proves the result.

**References**


