The Reciprocal Gutman Index (RGut), defined for a connected graph as vertex degree weighted sum of the reciprocal distances, i.e. \( \text{RGut}(G) = \sum_{u, v \in V(G)} \frac{d_G(u) d_G(v)}{d_G(u, v)} \). The RGut is a weighted version of the Harary index, just as the Gutman index is a weighted version of the Wiener index. In this paper, we present exact expression for the reciprocal Gutman index of Cartesian product and the wreath product of complete graphs with odd vertices in terms of other graph invariants including the reciprocal degree distance, Harary index, the first Zagreb index and the second Zagreb index.

KEYWORDS: Reciprocal Gutman index, Reciprocal degree distance, Cartesian Product, Wreath Product, Harary index, first and second Zagreb indices.
Zagreb index $M_2(G)$ of a graph $G$ are defined as, $M_1(G) = \sum_{u \in V(G)} [d_G(u) + d_G(v)]$ and $M_2(G) = \sum_{u \in V(G)} [d_G(u)d_G(v)]$ The Multiplicative Zagreb indices of graph operations have been obtained in [18].

In this paper, we found a new graph invariant named reciprocal Gutman index, which can be seen as a degree weight version of Harary index that is $RGut(G) = \frac{1}{2} \sum_{u \in V(G)} \frac{d_G(u)d_G(v)}{d_G(u,v)}$ And we obtain the reciprocal Gutman indices of Cartesian and wreath products of complete graphs with odd vertices.

**GUTMAN INDEX OF CARTESIAN PRODUCT OF COMPLETE GRAPHS**

In this section, we find the reciprocal Gutman index of the Cartesian product $G_1 \Box G_2$ of the complete graphs $G_1$ and $G_2$ with odd vertices. Let $V(G_1) = \{u_0, u_1, \ldots, u_{n-1}\}$, $V(G_2) = \{v_0, v_1, \ldots, v_{n_2-1}\}$ and let $w_{ij}$ be the vertex $(u_i, v_j)$ of $G_1 \Box G_2$.

The following lemma follows from the definition of the Cartesian product of the two graphs.

**Lemma 2.1**

Let $G_1$ and $G_2$ be two complete graphs. Let $w_{ij} = (u_i, v_j)$ and $w_{pq} = (u_p, v_q)$ be in $V(G_1 \Box G_2)$. Then $d_{G_1 \Box G_2}(w_{ij}, w_{pq}) = 2d_{G_1}(u_i, u_p)$ and $d_{G_1 \Box G_2}(w_{ij}) = d_{G_1}(u_i) + d_{G_2}(v_j)$.

**Theorem 2.2**

If $G_1$ and $G_2$ are two complete graphs with odd vertices $\mid V(G_1) \mid = n_1$ and $\mid V(G_2) \mid = n_2$, where $n_1, n_2 \geq 3$ then

$$RGut(G_1 \Box G_2) = \frac{1}{2} [H(G_1)M_1(G_2) + 4e(G_2)RDD(G_1) + 2n_1RGut(G_1) + 2H(G_2)M_1(G_1) + 4e(G_1)RDD(G_2) + 2n_1RGut(G_2) + n_2(n_2 - 1)RGut(G_1) + 4n_1(n_1 - 1)(n_2 - 1)e(G_2) + 4H(G_1)e(G_2)^2]$$

where $RGut(G), H(G), M_1(G)$ and $RDD(G)$ denote the reciprocal Gutman index, the Harary index, the first Zagreb index and the reciprocal degree distance of $G$ respectively.

**Proof**

Let $G = G_1 \Box G_2$. Then

$$RGut(G) = \frac{1}{2} \sum_{w_{ij}, w_{pq} \in V(G)} \frac{d_G(w_{ij})d_G(w_{pq})}{d_G(w_{ij}, w_{pq})}$$

$$= \frac{1}{2} \left\{ \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \frac{d_G(w_{ij})d_G(w_{ij})}{d_G(w_{ij}, w_{ij})} + \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \frac{d_G(w_{ij})d_G(w_{ij})}{d_G(w_{ij}, w_{ij})} + \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \frac{d_G(w_{ij})d_G(w_{ij})}{d_G(w_{ij}, w_{ij})} \right\}$$

$$= \frac{1}{2} \left( A + B + C \right)$$

where $A, B, C$ are the sums of the above terms in order. We shall calculate $A, B, C$ of (2.1) separately.

First we calculate,

$$A = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \frac{d_G(w_{ij})d_G(w_{ij})}{d_G(w_{ij}, w_{ij})}$$

For this we first compute

$$A' = \sum_{(i,p=0, i \ne p)} \frac{d_G(w_{ij})d_G(w_{ij})}{d_G(w_{ij}, w_{ij})}$$

$$= \sum_{i,p=0, i \ne p} \frac{d_G(u_i)d_G(v_j) + d_G(v_j)d_G(u_i)}{d_G(u_i, v_j)}$$

by the Lemma 2.1

$$= \sum_{i,p=0, i \ne p} \left( d_G(u_i)d_G(u_p) + d_G(v_j)d_G(v_p) \right) + \frac{1}{d_G(u_i, v_j)}$$

by the definitions of reciprocal Gutman index, reciprocal degree distance and Harary index of a graph.

\[ A = \sum_{j=0}^{n_2-1} \left[ 2RGut(G_1) + 2d_{G_1}(v_j)RDD(G_1) + 2d^2_{G_1}(v_j)H(G_1) \right] \]

Next we calculate

\[ B = \sum_{i=0}^{n_1-1} \sum_{j,q=0,j \neq q} d_G(w_{ij})d_G(w_{iq}) \]

For this we compute

\[ B' = \sum_{j,q=0,j \neq q} \frac{d_G(w_{ij})d_G(w_{iq})}{d_G(v_jv_q)} \]

by the definitions of reciprocal Gutman index, reciprocal degree distance and Harary index of a graph.

\[ B = \sum_{i=0}^{n_1-1} \sum_{j,q=0,j \neq q} \frac{d_G(w_{ij})d_G(w_{iq})}{d_G(v_jv_q)} \]

Next we compute

\[ C = \sum_{j,q=0,j \neq q} \sum_{i=0}^{n_1-1} \frac{d_G(w_{ij})d_G(w_{pq})}{d_G(v_jv_q)} \]

For this we compute

\[ C' = \sum_{i=0}^{n_1-1} \sum_{l,p=0,l \neq p} \frac{d_G(w_{ij})d_G(w_{pq})}{d_G(v_jv_q)} \]

by the definitions of reciprocal Gutman index and Harary index of a graph.

\[ C = \sum_{j,q=0,j \neq q} \sum_{i=0}^{n_1-1} \frac{d_G(w_{ij})d_G(w_{pq})}{d_G(v_jv_q)} \]
GUTMAN INDEX OF WREATH PRODUCT OF COMPLETE GRAPHS

In this section, we find the reciprocal Gutman index of the wreath product $G_1 \ast G_2$ of the complete graphs $G_1$ and $G_2$ with odd vertices. Let $V(G_1) = \{u_0, u_1, \ldots, u_{n_1 - 1}\}$, $V(G_2) = \{v_0, v_1, \ldots, v_{n_2 - 1}\}$, and let $w_{ij}$ be the vertex $(u_i, v_j)$ of $G_1 \ast G_2$.

The following lemma follows from the definition of the wreath product of the two graphs.

**Lemma 3.1**

Let $G_1$ and $G_2$ be two complete graphs. Let $w_{ij} = (u_i, v_j)$ and $w_{pq} = (u_p, v_q)$ be in $V(G_1 \ast G_2)$. Then

$$d_{G_1 \ast G_2}(w_{ij}, w_{pq}) = \begin{cases} d_{G_1}(u_i, u_p), & \text{if } i \neq p \\ 1, & \text{if } i \neq q \end{cases}$$

and $d_{G_1 \ast G_2}(w_{ij}) = |V(G_2)|d_{G_1}(u_i) + d_{G_2}(v_j)$.

**Theorem 3.2**

If $G_1$ and $G_2$ are two complete graphs with odd vertices $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$, then $n_1, n_2 \geq 3$, then

$$\begin{align*}
R\text{Gut}(G_1 \ast G_2) &= \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \sum_{d_{G_1}(u_i) = \lambda} \sum_{d_{G_2}(v_j) = \mu} 2d_1(u_i, v_j) \mu \lambda \\
&= \frac{1}{2} \left( n_1 n_2 - \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} d_{G_1}(u_i) d_{G_2}(v_j) \right)
\end{align*}$$

where $R\text{Gut}(G)$, $H(G)$, $M_1(G)$, $M_2(G)$ and $R\text{DD}(G)$ denote the reciprocal Gutman index, the Harary index, the first Zagreb index, the second Zagreb index and the reciprocal degree distance of $G$ respectively.

**Proof**

Let $G = G_1 \ast G_2$. Then

$$R\text{Gut}(G) = \frac{1}{2} \sum_{w_{ij}, w_{pq} \in V(G)} \frac{d_G(w_{ij})d_G(w_{pq})}{d_G(w_{ij}, w_{pq})}$$

$$= \frac{1}{2} \left( \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} d_{G_1}(u_i) d_{G_2}(v_j) \right)$$

where $A, B, C$ are the sums of the above terms in order. We shall calculate $A, B, C$ of (3.1) separately. First we calculate, $A = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \frac{d_{G}(u_i,v_j)}{d_{G}(u_i,v_j)}$

For this we first compute

$$A' = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \frac{d_{G_1}(u_i)d_{G_2}(v_j)}{d_{G_1}(u_i,v_j)}$$

by the Lemma 3.1

$$= 2n_2 R\text{Gut}(G_1) + 2n_2 d_2(G_1) R\text{DD}(G_1) + 2d_2(G_1) H(G_1)$$

by the definitions of reciprocal Gutman index, reciprocal degree distance and Harary index of a graph.
\[ A = \sum_{j=0}^{n_2-1} \left[ 2n_2^2R_{\text{Gut}}(G_1) + 2n_2d_{G_2}(v_j)R_{\text{DD}}(G_1) + 2d_{G_2}^2(v_j)H(G_1) \right] \]
\[ = 2n_2^2R_{\text{Gut}}(G_1) + 4n_2e(G_2)R_{\text{DD}}(G_1) + 2H(G_1)M_1(G_2) \]  
(3.2)

Next we calculate
\[ B = \sum_{i=0}^{n_1-1} \sum_{j=0, j \neq q}^{n_2-1} \frac{d_G(w_{ij})d_G(w_{ij})}{d_G(w_{ij})w_{ij}} \]

For this we compute
\[ B' = \sum_{j,q=0, j \neq q}^{n_2-1} \frac{d_G(w_{ij})d_G(w_{ij})}{d_G(w_{ij})w_{ij}} \]
\[ = \sum_{j,q=0}^{n_2-1} \left[ n_2d_{G_1}(u_i) + d_{G_2}(v_j) \right] \left[ n_2d_{G_1}(u_i) + d_{G_2}(v_j) \right] \]
\[ = \sum_{j,q=0}^{n_2-1} \left[ d_{G_2}(v_j)d_{G_2}(v_j) + n_2d_{G_1}(u_i) \left( \sum_{j,q=0,j \neq q}^{n_2-1} d_{G_2}(v_j) + d_{G_2}(v_j) \right) + n_2d_{G_1}^2(u_i) \sum_{j,q=0,j \neq q}^{n_2-1} 1 \right] \]
\[ = 2n_2^2e(G_2)d_{G_1}^2(u_i) + n_2d_{G_1}(u_i)M_1(G_2) + M_2(G_2) \]
by the definitions of the first and second Zagreb indices of a graph.
\[ \therefore B = \sum_{i=0}^{n_1-1} \sum_{j,p=0, j \neq p}^{n_2-1} \frac{d_G(w_{ij})d_G(w_{ij})}{d_G(w_{ij})w_{ij}} \]

For this we compute
\[ C' = \sum_{j,q=0, j \neq q}^{n_2-1} \frac{d_G(w_{ij})d_G(w_{ij})}{d_G(w_{ij})w_{ij}} \]
\[ = \sum_{j,q=0}^{n_2-1} \frac{n_2d_{G_1}(u_i)d_{G_1}(u_p) + d_{G_2}(v_j)d_{G_2}(v_j)}{d_{G_1}(u_i,u_p)} \]
\[ = n_2^2d_{G_2}(v_j) \left[ d_{G_1}(u_i) + d_{G_1}(u_p) \right] + 2n_2(n_2-1)e(G_2) \left[ d_{G_1}(u_i) + d_{G_1}(u_p) \right] \]
+ \left( 4e(G_2)^2 - M_1(G_2) \right) \frac{1}{d_{G_1}(u_i,u_p)} \]
\[ \therefore C = \sum_{j,p=0, j \neq p}^{n_1-1} \left[ n_2^2d_{G_2}(v_j) \left[ d_{G_1}(u_i) + d_{G_1}(u_p) \right] + 2n_2(n_2-1)e(G_2) \left[ d_{G_1}(u_i) + d_{G_1}(u_p) \right] \right] 
+ \left( 4e(G_2)^2 - M_1(G_2) \right) \frac{1}{d_{G_1}(u_i,u_p)} \]
\[ = n_2^2(n_2-1)R_{\text{Gut}}(G_1) + 4n_2(n_2-1)e(G_2)R_{\text{DD}}(G_1) + 2(4e(G_2)^2 - M_1(G_2))H(G_1) \]  
(3.4)

Using 3.2,3.3,3.4 in 3.1 we get
\[ RG_{\text{Gut}}(G_1, G_2) = n_2^2 e(G_2)M_1(G_1) + 2n_2e(G_1)M_2(G_2) + n_1M_2(G_2) + n_2^3R_{\text{Gut}}(G_1) + 2n_2R_{\text{DD}}(G_1)e(G_2) \\
+ H(G_1)M_1(G_2) + n_3^3(n_2 - 1)R_{\text{Gut}}(G_1) + 2n_3(n_2 - 1)e(G_2)R_{\text{DD}}(G_1) \\
+ H(G_1)[4e(G_2)^2 - M_1(G_2)] \]

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