ABSTRACT


KEYWORDS: Generalised nearly Lorentzian Sasakian manifold, generalised almost Lorentzian Sasakian manifold, generalised Lorentzian Co-symplectic manifolds, semi-symmetric metric F-connection.

INTRODUCTION

An $n (=2m+1)$ dimensional differentiable manifold $M_n$, which admits a tensor field $F$ of type $(1, 1)$, two contravariant vector fields $T_1$ and $T_2$, two covariant vector fields $A_1$ and $A_2$ and a Lorentzian metric $g$, satisfying for arbitrary vector fields $X, Y, Z, ...$

$$\overline{X} = -X - A_1(X)T_1 - A_2(X)T_2, \overline{T_1} = 0, \overline{T_2} = 0, \ A_1(T_1) = -1, \ A_2(T_2) = -1, \ \overline{X} \equiv FX, \ A_1(\overline{X}) = 0,$$

$$A_2(\overline{X}) = 0, \ \text{rank} \ F = n-2$$

(1.2) $g (\overline{X}, \overline{Y}) = g (X, Y) + A_1(X)A_1(Y) + A_2(X)A_2(Y)$, where $A_1(X) = g(X, T_1), \ A_2(X) = g(X, T_2)$

Then $M_n$ is called a generalised Lorentzian contact manifold (generalised L-contact manifold) and the structure $(F, T_1, T_2, A_1, A_2, g)$ is known as generalised Lorentzian contact structure.

On a generalised L-contact manifold, we have

(1.3) $\overline{F}(X, Y) + \overline{F}(Y, X) = 0 \quad \text{(b) } \overline{F}(\overline{X}, \overline{Y}) = \overline{F}(X, Y)$

(c) $\overline{(D_x F)(Y, T_1)} = -(D_x A_1)(Y) \quad \text{(d) } \overline{(D_x F)(Y, T_2)} = -(D_x A_2)(Y)$

(1.4) $\overline{(D_x F)(\overline{Y}, Z)} - (D_x F)(Y, \overline{Z}) + A_1(Y)(D_x A_1)(Z) + A_2(Y)(D_x A_2)(Z) + A_1(Z)(D_x A_1)(Y) + A_2(Z)(D_x A_2)(Y) = 0$

$\overline{(D_x F)(\overline{Y}, \overline{Z})} = (D_x F)(\overline{Y}, \overline{Z})$

(c) $\overline{(D_x F)(\overline{Y}, \overline{Z})} + (D_x F)(Y, Z) + A_1(Y)(D_x A_1)(\overline{Z}) + A_2(Y)(D_x A_2)(\overline{Z}) - A_1(Z)(D_x A_1)(\overline{Y}) - A_2(Z)(D_x A_2)(\overline{Y}) = 0$

$\overline{(D_x F)(\overline{Y}, \overline{Z})} + (D_x F)(\overline{Y}, \overline{Z}) = 0$

Where $D$ is the Riemannian connection on $M_n$.
A generalised L-contact manifold is called a generalised Lorentzian Sasakian manifold if

\[(1.5) \ (a) \quad 2(D_x F)(Y) - \{A_1(Y) + A_2(Y)\} \overline{X} - g(\overline{X}, \overline{Y})(T_1 + T_2) = 0 \Leftrightarrow \]
\[\quad \{B_1(Y) + B_2(Y)\} g(\overline{X}, \overline{Y}) - \{A_1(Z) + A_2(Z)\} g(\overline{X}, \overline{Y}) = 0 \]
\[\quad \{C_1(Y) + C_2(Y)\} g(\overline{X}, \overline{Y}) = \{D_1(Y) + D_2(Y)\} g(\overline{X}, \overline{Y}) = 0 \]
\[\quad 2D_x T_1 = \overline{X} - T_2, \quad 2D_x T_2 = \overline{X} - T_1 \]

From which, we get

\[(1.6) \ (a) \quad 2(D_x A_1)(\overline{Y}) = 2(D_x A_2)(\overline{Y}) = g(\overline{X}, \overline{Y}) \Leftrightarrow \]
\[\quad 2(D_x A_1)(Y) + A_2(Y) = 2(D_x A_2)(Y) + A_1(Y) = \gamma F(Y, Y) \Leftrightarrow \]

And

\[(1.7) \ (a) \quad 2(D_x \gamma F)(\overline{Y}, Z) + \{A_1(Z) + A_2(Z)\} \gamma F(Y, Y) = 0 \]
\[\quad \{B_1(Y) + B_2(Y)\} \gamma F(\overline{X}, \overline{Z}) - \{A_1(Z) + A_2(Z)\} \gamma F(\overline{X}, \overline{Z}) = 0 \]
\[\quad \{C_1(Y) + C_2(Y)\} \gamma F(\overline{X}, \overline{Z}) = \{D_1(Y) + D_2(Y)\} \gamma F(\overline{X}, \overline{Z}) = 0 \]

Nijenhuis tensor in a generalised L-contact manifold is given as

\[(1.8) \quad \gamma N(Y, Z) = (D_x \gamma F)(Y, Z) + (D_y \gamma F)(Z, X) + (D_x \gamma F)(Y, \overline{Z}) + (D_y \gamma F)(\overline{Z}, X) \]

Where \( \gamma N(Y, Z) \equiv g(N(Y, Y), Z) \)

**GENERALISED NEARLY AND ALMOST LORENTZIAN SASAKIAN MANIFOLDS**

A generalised L-contact manifold will be called a generalised nearly Lorentzian Sasakian manifold if

\[(2.1) \quad 2(D_x \gamma F)(Y, Z) + \{A_1(Y) + A_2(Y)\} \gamma F(\overline{X}, \overline{Z}) - \{A_1(Z) + A_2(Z)\} \gamma F(\overline{X}, \overline{Y}) = 0 \]
\[\quad 2(D_y \gamma F)(Y, Z) + \{A_1(Y) + A_2(Y)\} \gamma F(\overline{X}, \overline{Z}) - \{A_1(Z) + A_2(Z)\} \gamma F(\overline{X}, \overline{Y}) = 0 \]

The equation of a generalised nearly Lorentzian Sasakian manifold can be modified as

\[(2.2) \ (a) \quad 2(D_x F)(Y, Z) + \{A_1(Y) + A_2(Y)\} \gamma F(\overline{X}, \overline{Y}) - \{A_1(Z) + A_2(Z)\} \gamma F(\overline{X}, \overline{Y}) = 0 \]
\[\quad \{B_1(Y) + B_2(Y)\} \gamma F(\overline{X}, \overline{Z}) + \{A_1(X) + A_2(X)\} \gamma F(\overline{X}, \overline{Z}) - \{A_1(Z) + A_2(Z)\} \gamma F(\overline{X}, \overline{Z}) = 0 \]
\[\quad \{C_1(Y) + C_2(Y)\} \gamma F(\overline{X}, \overline{Z}) = \{D_1(Y) + D_2(Y)\} \gamma F(\overline{X}, \overline{Z}) = 0 \]

This gives

\[(2.3) \ (a) \quad 2(D_x F)(\overline{Y}) + 2(D_y F)(X, Z) + \{A_1(X) + A_2(X)\} \gamma F(\overline{X}, \overline{Y}) + \{A_1(Z) + A_2(Z)\} \gamma F(\overline{X}, \overline{Y}) = 0 \]
\[\quad 2(D_x F)(\overline{Z}) - 2(D_y F)(X, Z) + \{A_1(X) + A_2(X)\} \gamma F(\overline{X}, \overline{Z}) + \{A_1(Z) + A_2(Z)\} \gamma F(\overline{X}, \overline{Z}) = 0 \]

\[(2.4) \ (a) \quad 2(D_x F)(\overline{Y}) + 2(D_y F)(X, Z) + \{A_1(X) + A_2(X)\} \gamma F(\overline{X}, \overline{Y}) = 0 \]

A generalised L-contact manifold will be called a generalised almost Lorentzian Sasakian manifold if

\[(2.6) \quad (D_X F)(Y, Z) + (D_X F)(Z, X) + (D_X F)(X, Y) = 0\]

**GENERALISED LORENTZIAN CO SYMPLECTIC MANIFOLDS**

A generalised L-contact manifold will be called a generalised Lorentzian Co-symplectic manifold if

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\[(2.5) \quad (D_X F)Y + (D_X F)X - A_1(Y)(D_X T_1) - A_2(X)(D_X T_2) - (D_X A_1)(\overline{Y})T_1 - (D_X A_2)(\overline{Y})T_2 = 0 \iff \]

\[
(\begin{align*}
(D_X F)(Y, Z) + A_1(Y)(D_X A_1)(\overline{Z}) + A_2(X)(D_X A_2)(\overline{Z}) - A_1(Z)(D_X A_1)(\overline{Y}) - A_2(Z)(D_X A_2)(\overline{Y}) &= 0
\end{align*}
\]

Therefore, a generalised Lorentzian Co-symplectic manifold is a generalised Lorentzian Sasakian manifold if

\[(3.2) \quad (D_X Y)(\overline{Y}) = 2(D_X A_2)(\overline{Y}) = g(\overline{X}, \overline{Y}) \iff \]

\[2(D_X A_1)(\overline{Y}) = 2(D_X A_2)(\overline{Y}) + A_1(Y) = \overline{F}(X, Y) \iff (c) \quad 2D_X T_1 = \overline{X} - T_2, \quad 2D_X T_2 = \overline{X} - T_1\]

A generalised L-contact manifold will be called a generalised nearly Lorentzian Co-symplectic manifold if

\[(3.3) \quad (D_X F)(Y, Z) + A_1(Y)(D_X A_1)(\overline{Z}) + A_2(X)(D_X A_2)(\overline{Z}) - A_1(Z)(D_X A_1)(\overline{Y}) - A_2(Z)(D_X A_2)(\overline{Y}) = 0\]

\[= (D_Y F)(Z, X) + A_1(Z)(D_Y A_1)(\overline{X}) + A_2(X)(D_Y A_2)(\overline{X}) - A_1(X)(D_Y A_1)(\overline{Z}) - A_2(X)(D_Y A_2)(\overline{Z})\]

\[= (D_Z F)(X, Y) + A_1(X)(D_Z A_1)(\overline{Y}) + A_2(Y)(D_Z A_2)(\overline{Y}) - A_1(Y)(D_Z A_1)(\overline{X}) - A_2(Y)(D_Z A_2)(\overline{X})\]

It is clear that a generalised nearly Lorentzian Sasakian manifold is a generalised nearly Lorentzian Co-symplectic manifold, in which

\[(3.4) \quad (D_X A_1)(\overline{Y}) = 2(D_X A_2)(\overline{Y}) \iff \]

\[2(D_X A_1)(Y) + A_1(Y) = 2(D_X A_2)(Y) + A_2(Y) = \overline{F}(X, Y) \iff (c) \quad 2D_X T_1 = \overline{X} - T_2, \quad 2D_X T_2 = \overline{X} - T_1\]

A generalised L-contact manifold will be called a generalised almost Lorentzian Co-symplectic manifold if

\[(3.5) \quad (D_X F)(Y, Z) + (D_X F)(Z, X) + (D_X F)(X, Y) - A_1(Y)(D_X A_1)(\overline{Z}) - (D_X A_1)(\overline{Y}) = 0\]

\[= A_2(X)(D_Y A_2)(\overline{Z}) - (D_Y A_2)(\overline{X}) - A_1(Y)(D_Y A_1)(\overline{X}) - (D_Y A_1)(\overline{Z}) - A_2(Y)(D_Y A_2)(\overline{X}) - (D_Y A_2)(\overline{Z}) - A_1(Z)(D_Z A_1)(\overline{Y}) - A_2(Z)(D_Z A_2)(\overline{Y})\]

Therefore, a generalised almost Lorentzian Co-symplectic manifold is a generalised almost Lorentzian Sasakian manifold if

\[(3.6) \quad (D_X A_1)(\overline{Y}) = 2(D_X A_2)(\overline{Y}) \iff \]

\[2(D_X A_1)(Y) + A_1(Y) = 2(D_X A_2)(Y) + A_2(Y) = \overline{F}(X, Y) \iff (c) \quad 2D_X T_1 = \overline{X} - T_2, \quad 2D_X T_2 = \overline{X} - T_1\]
(b) \(2(D_XA_1)(Y) + A_2(Y) = 2(D_XA_2)(Y) + A_1(Y) = ^{\bar{\nabla}}F(X,Y) \iff 2D_XT_1 = \bar{X} - T_1, \quad 2D_XT_2 = \bar{X} - T_2\)

**COMPLETELY INTEGRABLE MANIFOLDS**

Barring \(X, Y, Z\) in (1.8) and using equations (2.1), (1.4) (b), we see that a generalised nearly Lorentzian Sasakian manifold is completely integrable if

\[(4.1) \ (D_X^F)(\bar{X}, \bar{Z}) = (D_X^F)(\bar{X}, \bar{Z})\]

Barring \(X, Y, Z\) in (1.8) and using equations (2.6), (1.4) (b), we see that a generalised almost Lorentzian Sasakian manifold is completely integrable if

\[(4.2) \ (D_X^F)(\bar{X}, \bar{Y}) = 0.\]

**SEMI_SYMMETRIC METRIC F-CONNECTION IN A GENERALISED LORENTZIAN SASAKIAN MANIFOLD**

Let \(M_{2m-1} \) be submanifold of \(M_{2m+1}\) and let \(c : M_{2m-1} \to M_{2m+1}\) be the inclusion map such that

\[d \in M_{2m-1} \to cd \in M_{2m+1},\]

Where \(c\) induces a linear transformation (Jacobian map) \(J : T'_{2m-1} \to T'_{2m+1}.\)

\(T'_{2m-1}\) is a tangent space to \(M_{2m-1}\) at point \(d\) and \(T'_{2m+1}\) is a tangent space to \(M_{2m+1}\) at point \(cd\) such that \(\hat{X}\) in \(M_{2m-1}\) at \(d\) \(\to \hat{J}(\hat{X})\) in \(M_{2m+1}\) at \(cd\)

Let \(\hat{g}\) be the induced Lorentzian metric in \(M_{2m-1}\). Then we have

\[(5.1) \ \hat{g}(\hat{X}, \hat{Y}) = ((g(J\hat{X}, J\hat{Y}))b)\]

We now suppose that a semi-symmetric metric F-connection \(B\) in a generalised Lorentzian Sasakian manifold is given by

\[(5.2) \ 2B_XY = 2D_XY + A_1(Y)FX + A_2(Y)FX - g(FX,Y)T_1 - g(FX,Y)T_2 + 2(A_1(X) + A_2(X))FY,\]

Where \(X\) and \(Y\) are arbitrary vector fields of \(M_{2m+1}\). If

\[(5.3) \ (a) \ T_1 = Jt_1 + \rho_1M + \sigma_1N \quad \text{and} \]

\[(b) \ T_2 = Jt_2 + \rho_2M + \sigma_2N\]

Where \(t_1\) and \(t_2\) are \(C^\infty\) vector fields in \(M_{2m-1}\) and \(M\) and \(N\) are unit normal vectors to \(M_{2m-1}\).

Denoting by \(\hat{D}\) the connection induced on the submanifold from \(D\), we have Gauss equation

\[(5.4) \ D_{\hat{X}}\hat{Y} = J(D_X\bar{Y}) + h(\hat{X}, \bar{Y})M + k(\hat{X}, \bar{Y})N\]

Where \(h\) and \(k\) are symmetric bilinear functions in \(M_{2m-1}\). Similarly we have

\[(5.5) \ B_{\hat{X}}\hat{Y} = J(B_X\bar{Y}) + p(\hat{X}, \bar{Y})M + q(\hat{X}, \bar{Y})N,\]

Where \(\hat{B}\) is the connection induced on the submanifold from \(B\) and \(p\) and \(q\) are symmetric bilinear functions in \(M_{2m-1}\).

In consequence of (5.2), we have
(5.6) \[ 2B_{jX}Y = 2D_{jX}Y + A_1(JY)F\hat{X} + A_2(JY)F\hat{X} - g(JF\hat{X},JY)T_1 - g(JF\hat{X},JY)T_2 + 2\{A_1(J\hat{X})F\hat{Y} + A_2(J\hat{X})F\hat{Y}\} \]

Using (5.4), (5.5) and (5.6), we get

(5.7) \[ 2J(\hat{B}_X\tilde{Y}) + 2p(\tilde{X},\tilde{Y})M + 2q(\tilde{X},\tilde{Y})N = 2J(\hat{D}_X\tilde{Y}) + 2h(\tilde{X},\tilde{Y})M + 2k(\tilde{X},\tilde{Y})N + A_1(JY)F\hat{X} + A_2(JY)F\hat{Y} - g(JF\hat{X},J\tilde{Y})T_1 - g(JF\hat{X},J\tilde{Y})T_2 + 2\{A_1(J\hat{X})F\hat{Y} + A_2(J\hat{X})F\hat{Y}\} \]

Using (5.3) (a) and (5.3) (b), we obtain

(5.8) \[ 2J(\hat{B}_X\tilde{Y}) + 2p(\tilde{X},\tilde{Y})M + 2q(\tilde{X},\tilde{Y})N = 2J(\hat{D}_X\tilde{Y}) + 2h(\tilde{X},\tilde{Y})M + 2k(\tilde{X},\tilde{Y})N + a_1(\tilde{Y})JF\hat{X} + a_2(\tilde{Y})JF\hat{Y} - g(F\hat{X},\tilde{Y})(Jt_1 + \rho_1M + \sigma_1N) - g(F\hat{X},\tilde{Y})(Jt_2 + \rho_2M + \sigma_2N) + 2\{a_1(\tilde{X})F\hat{Y} + a_2(\tilde{X})F\hat{Y}\} \]

Where \( \bar{g}(\tilde{Y},t_1) \equiv a_1(\tilde{Y}) \) and \( \bar{g}(\tilde{Y},t_2) \equiv a_2(\tilde{Y}) \)

Which implies

(5.9) \[ 2B_X\tilde{Y} = 2\hat{D}_X\tilde{Y} + a_1(\tilde{Y})F\hat{X} + a_2(\tilde{Y})F\hat{X} - \bar{g}(F\hat{X},\tilde{Y})t_1 - \bar{g}(F\hat{X},\tilde{Y})t_2 + 2\{a_1(\tilde{X})F\hat{Y} + a_2(\tilde{X})F\hat{Y}\} \]

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(5.10) (a) \[ 2p(\tilde{X},\tilde{Y}) = 2h(\tilde{X},\tilde{Y}) - \rho_1\bar{g}(F\hat{X},\tilde{Y}) - \rho_2\bar{g}(F\hat{X},\tilde{Y}) \]

(b) \[ 2q(\tilde{X},\tilde{Y}) = 2k(\tilde{X},\tilde{Y}) - \sigma_1\bar{g}(F\hat{X},\tilde{Y}) - \sigma_2\bar{g}(F\hat{X},\tilde{Y}) \]

Thus we have

**Theorem 5.1** The connection induced on a submanifold of a generalised Lorentzian Sasakian manifold with a semi-symmetric metric F-connection with respect to unit normal vectors \( M \) and \( N \) is also semi-symmetric metric F-connection iff (5.10) holds.

**REFERENCES**


